Simple Linear Regression

Grinnell College

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Warm-up

Suppose from a population of male Adelie penguins we take measurements on flipper length and find the following statistics:

$$\overline{x} = 190\text{mm}, \qquad \hat{\sigma} = 6.54\text{mm}$$

If a particular penguin had a standardized flipper length of $z = -0.5$, what was the length of his flipper in millimeters?

Z-scores and Correlation

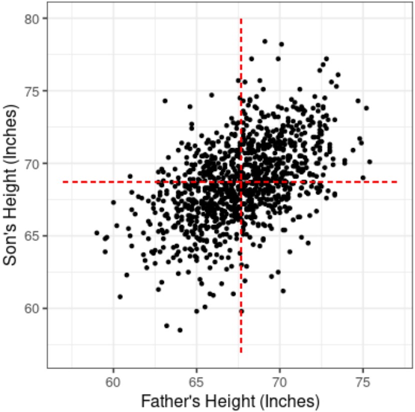
Recall that:

- **Z-scores** or **standardized scores** relate each observation to the mean and standard deviation of the variable
  - $z = 0$ corresponds to the average and $z = 1$ corresponds to one standard deviation
- **Correlation** specifies the *linear* relationship between two quantitative variables

Pearson's Height Data

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

Father	Son
65.0	59.8
63.3	63.2
65.0	63.3
65.8	62.8
61.1	64.3
63.0	64.2
⋮	⋮



Regression towards the mean

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

The correlation coefficient tells us how much "regression" we expect to observe in terms of standardized values. Letting X and Y represent father and son, respectively, we have:

$$z_Y = r \times z_X$$

If the father is one and a half standard deviations above average ($z_F = 1.5$), and the correlation between heights is 0.501, we have:

$$\begin{aligned} z_Y &= r \times z_X \\ &= 0.501 \times 1.5 \\ &= 0.752 \end{aligned}$$

Correlation and Prediction

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

From here, we can back substitute the value for z_Y to get our unstandardized predictions:

$$
\begin{aligned}
z_Y &= 0.752 \\
\left(\frac{\hat{y} - 68.68}{2.81}\right) &= 0.752 \\
\hat{y} &= 0.752 \times 2.81 + 68.68 \\
\hat{y} &= 70.793
\end{aligned}
$$

Where $\hat{y}$ represents our best guess for y, given a value for x

Regression Line

1= The relationship $z_y = r \times z_x$ can always be manipulated to rewrite the relationship between the variables X and y so they fit the formula

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$

See that

$$z_y = r \times z_x$$

$$\frac{y - \bar{y}}{\hat{\sigma}_y} = r\left(\frac{x - \bar{x}}{\hat{\sigma}_x}\right)$$

$$y - \bar{y} = r\frac{\hat{\sigma}_y}{\hat{\sigma}_x}(x - \bar{x})$$

$$y = r\frac{\hat{\sigma}_y}{\hat{\sigma}_x}x - r\frac{\hat{\sigma}_y}{\hat{\sigma}_x}\bar{x} + \bar{y}$$

$$y = \underbrace{\left(r\frac{\hat{\sigma}_y}{\hat{\sigma}_x}\right)}_{\beta_1}x + \underbrace{\left(\bar{y} - r\frac{\hat{\sigma}_y}{\hat{\sigma}_x}\bar{x}\right)}_{\beta_0}$$

Regression Line

The relationship $z_y = r \times z_x$ can always be manipulated to rewrite the relationship between the variables X and y so they fit the formula

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$

We interpret these as follows:

- $\hat{\beta}_0$ represents the *intercept*, or the estimated value of y when $X = 0$
- $\hat{\beta}_1$ represents the *slope*, indicating the magnitude of change in y given a unit change in X

Regression Line from Z Scores

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

Note that $z_F = 1.5$ corresponds to $X = 71.79$

$$z_S = r \times z_F$$
$$\left(\frac{\hat{y} - 68.68}{2.81}\right) = r \times \left(\frac{X - 67.68}{2.74}\right)$$
$$\hat{y} = 33.9 + 0.514X$$

Where $\hat{y}$ represents our best guess for y, given a value for X

Predictions

The formula for the regression line

$$\hat{y} = \beta_0 + X\beta_1$$

can be expressed in terms our our original variables and what we wish to predict

$$\widehat{\text{Son's Height}} = 33.9 + 0.514 \times \text{Father's Height}$$

From this, there are a few things about lines we can observe:

- Using this line, *given* the Father's height, we can predict the son's height using this line by plugging in a value for the father's height
- "For each 1 inch change in Father's height, we expect to see a 0.51 inch change in Son's height"
- Intercept interpretation

Linear Model in R

Creating linear models in R is simple; the `lm()` function creates a *linear model* that requires a *formula* component, `Son ~ Father` and a data argument, specifying the dataset containing the variables

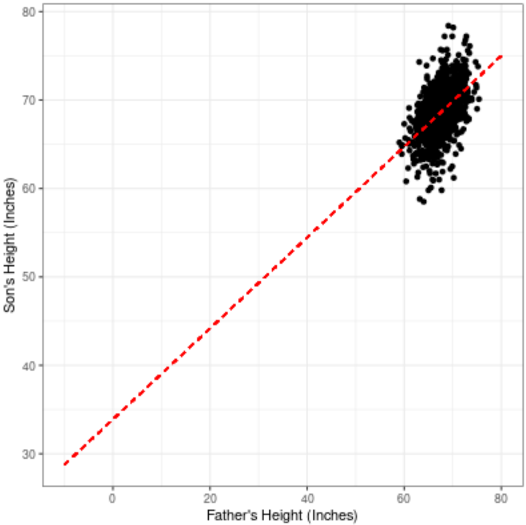
```
> lm(formula = Son ~ Father, data = dat)

Coefficients:
(Intercept)       Father
     33.893        0.514
```

The output gives us the intercept along with a value for the slope

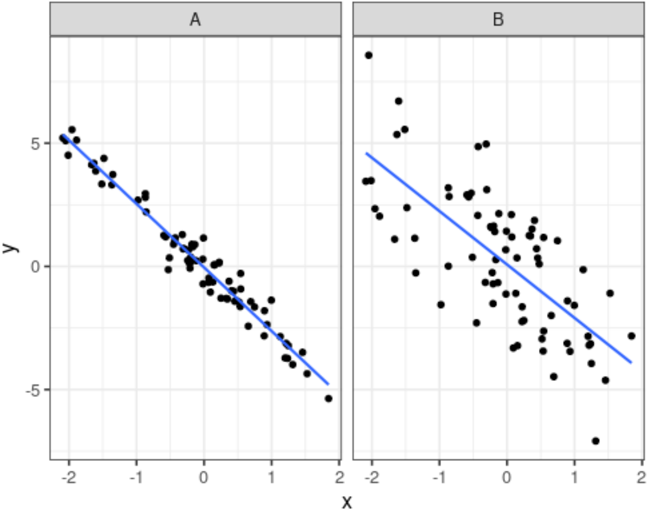
Intercept Interpretation/Extrapolation

$$\widehat{\text{Son's Height}} = 33.9 + 0.51 \times \text{Father's Height}$$



Assessing Quality of Fit

"How much variability is left once I have selected my prediction on the line?"



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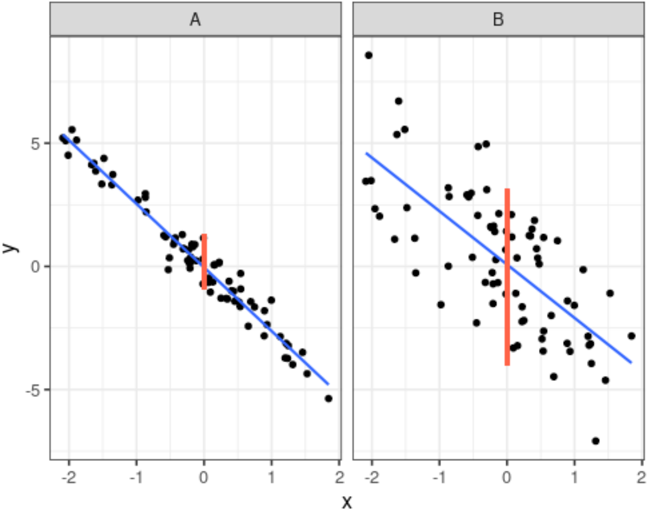


Total Sum of Squares

If we had an outcome y and no predictor variable x, our best guess for an estimate of y would simply by the mean, $\overline{y}$

From this, we get a sense of the *total variance* by taking the *sum of squares*:

$$\text{Total Sum of Squares} = \sum_{i=1}^{n} (y_i - \overline{y})^2$$

We can think of this as our baseline: this is how much variability we see with no other predictors

Regression Sum of Squares

Now assume for each y_i we used a variable x_i, along with their correlation, to create an estimated value $\hat{y}_i$, with

$$\hat{y}_i = \beta_0 + \beta_1 x_i$$

We could then ask ourselves: how much variability is left once I have used my predictor to make $\hat{y}_i$? This gives us the *residual sum of squares*:

$$\text{Residual Sum of Squares} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Coefficient of Determination

Now consider the ratio of variance explained in model against variance without model:

$$\frac{\text{Residual SS (SSR)}}{\text{Total SS (SST)}} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}$$

If our model is no better than guessing the average (i.e., if $\hat{y} = \overline{y}$), this ratio would be 1; if we are able to perfectly predict each value y_i, this ratio would be 0

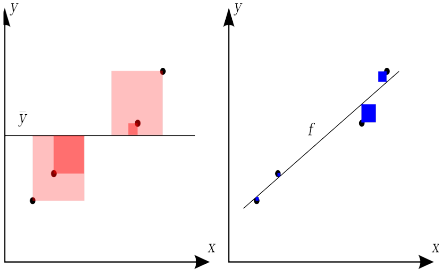
Our **coefficient of determination** or R^2 (R-squared) is defined as

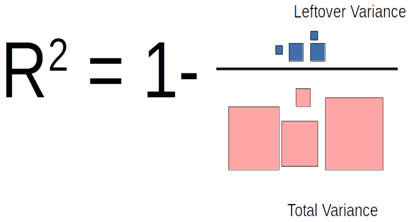
$$R^2 = 1 - \frac{SSR}{SST}$$

Somewhat surprisingly, in the case with a single predictor variable we have that the coefficient of determination is simply the squared correlation

$$R^2 = r^2$$

$$\frac{\text{Residual SS (SSR)}}{\text{Total SS (SST)}} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}$$





Review

We should be able to

- Describe how correlation and regression related
- Be able to predict an outcome, given a predictor
- Interpret the slope and intercept (if applicable)
- Assess the quality of a fitted line