# Residuals

Grinnell College

May 1, 2026

# Review

Below is a model of chicken weight in days since birth according to one of four separate diets

```
> lm(weight ~ Time + Diet, ChickWeight) %>% summary()

            Estimate Std. Error t value             Pr(>|t|)
(Intercept)   10.924      3.361    3.25               0.0012 **
Time           8.750      0.222   39.45 < 0.0000000000000002 ***
Diet2         16.166      4.086    3.96      0.00008556049098 ***
Diet3         36.499      4.086    8.93 < 0.0000000000000002 ***
Diet4         30.233      4.107    7.36      0.00000000000064 ***

Multiple R-squared:  0.745, Adjusted R-squared:  0.744
F-statistic:  419 on 4 and 573 DF,  p-value: <0.0000000000000002
```

1. Write out the equation for this linear model
2. What proportion of the total variability in chick weight is described by this model?
3. Does there appear to be a statistically significant difference between Diet 1 and Diet 3 at the $\alpha = 0.05$ level?
4. After 10 days, what is the predicted difference in weight between a chicken on Diet 2 and Diet 3?

# A few comments

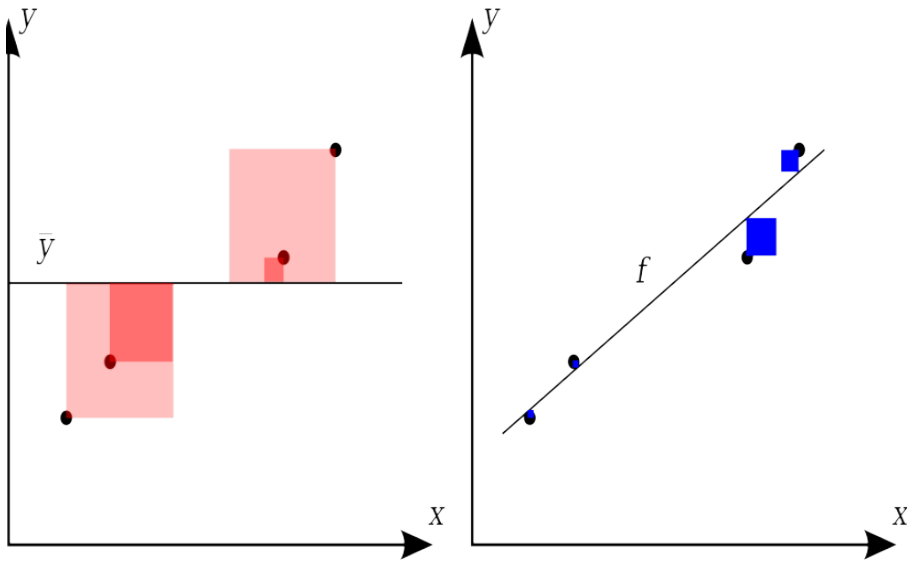
While we noted many similarities between ANOVA and regression Wednesday, we didn't review the metric for overall model fit

Where ANOVA has the F statistic, the ratio of explained to unexplained variance, regression used "R-squared", giving the ratio of unexplained variance to total variance

Where both the F statistic and $R^2$ are ratios, $R^2$ is a proper proportion (part / whole). As such, it necessarily takes values between 0 (no variance explained) and 1 (all variance explained)

Rather than being associated with a test, $R^2$ simply gives us a sense of model fit

# R Squared




$$R^2 = 1 - \frac{\text{Leftover Variance}}{\text{Total Variance}}$$

# A few extra comments

- $R^2$ *only* assess variance explained; it does not make any statements about the relationship between covariates and our outcome
- It is possible to have strong evidence of a relationship without necessarily explaining a lot of variation
- Where "Multiple R-squared" reports the ratio directly, "Adjusted R-squared" tries to qualify this with degrees of freedom based on sample size $n$ and number of predictors $k$:

$$R^2_{adj} = 1 - \left(\frac{n-1}{n-1-k}\right)(1-R^2)$$

- All we need to know about adjusted $R^2$ is that it's asking the question: did adding an extra predictor meaningfully reduce variance?

# Today

- Regression is a *model* posits linear relationship between dependent variable $y$ and independent variable $X$ of the form

$$y = \beta_0 + \beta_1 X + \epsilon$$

- Expand this to include combinations of independent variables, both qualitiative and quantitative
- Coefficients can represent one of two things:
  - Changes in slope (quantitative variables)
  - Changes in intercept (categorical variables)
- Today our focus is on the error term $\epsilon$ (epsilon)

# Error Terms

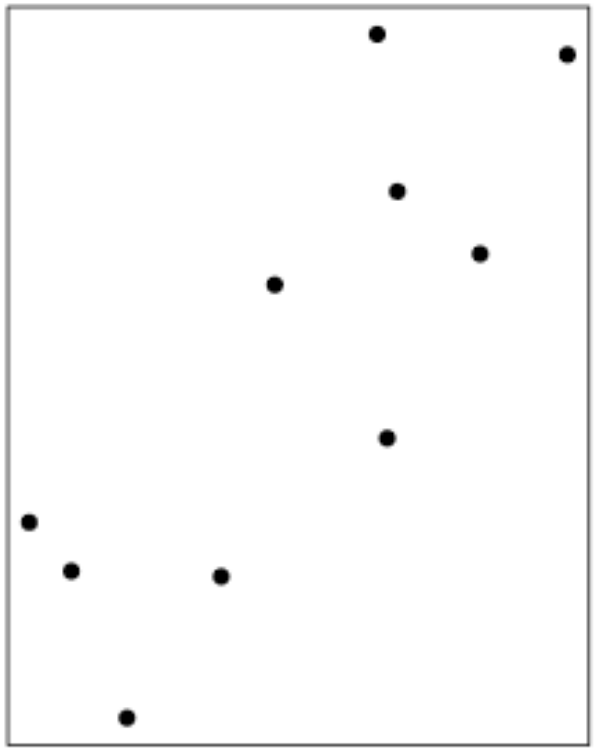
$$y = \beta_0 + \beta_1 X + \epsilon$$

Assumptions:

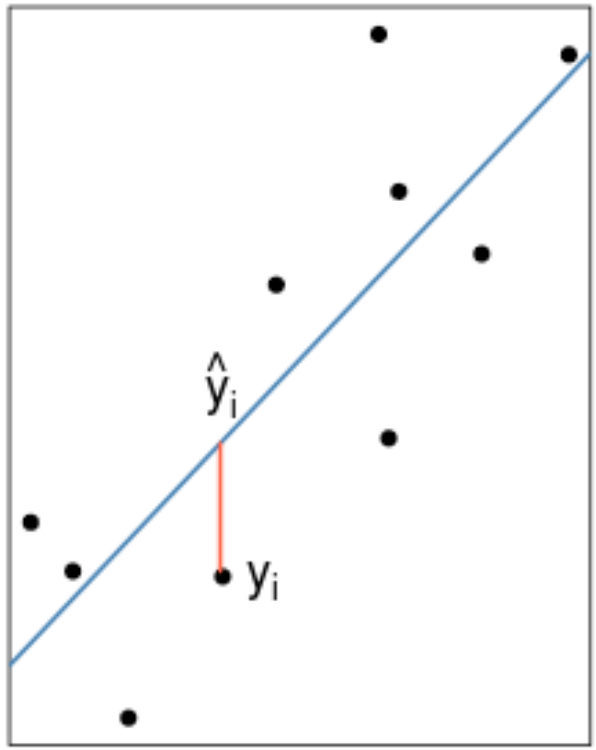
- Linear relationship between X and y
- Error term is normally distributed, $\epsilon \sim N(0, \sigma)$
- Variability in error should be the same for all values of $X$

Analyzing the error terms gives us a way to test the assumptions of our model

**Collection of (x, y) points**



**Fitted line with residual**

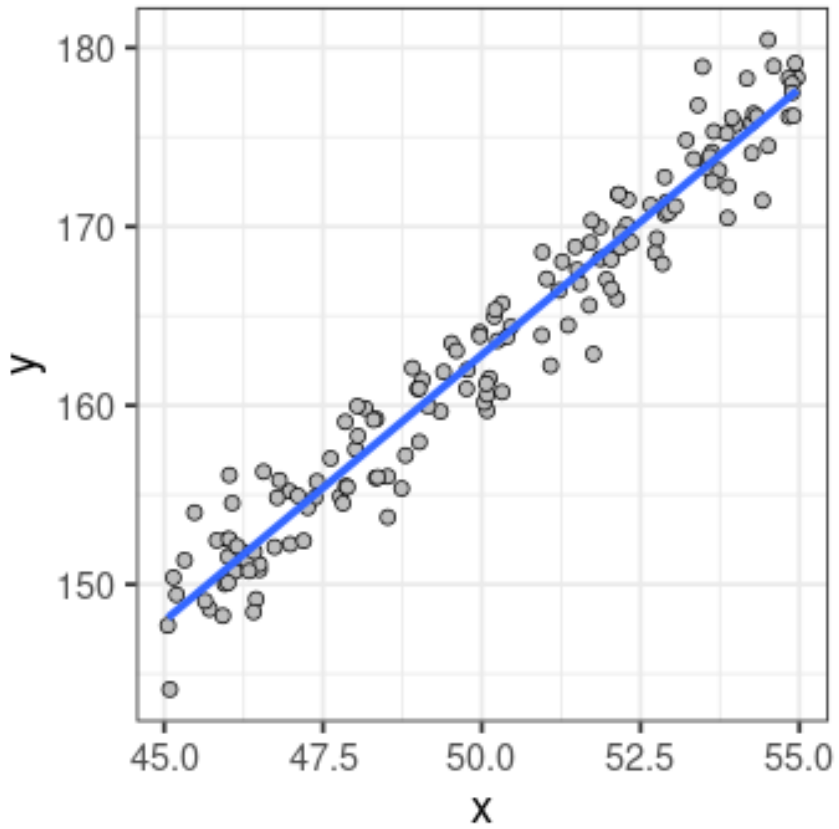


# Part 1: Checking Assumptions

# Residuals and assumptions

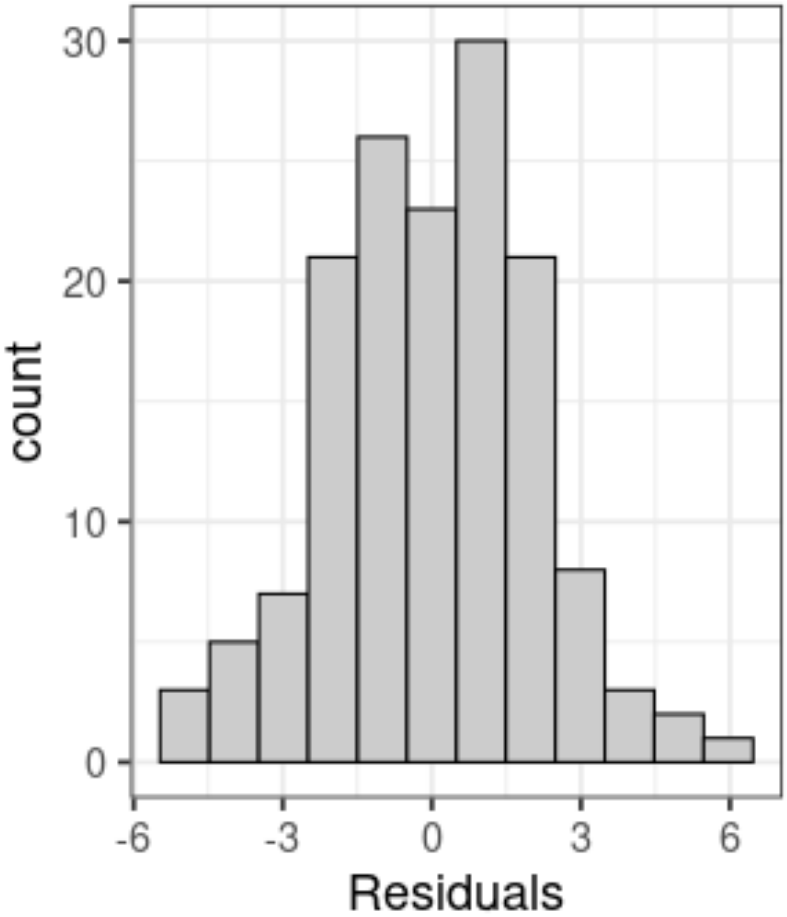
Three common ways to investigate residuals visually:

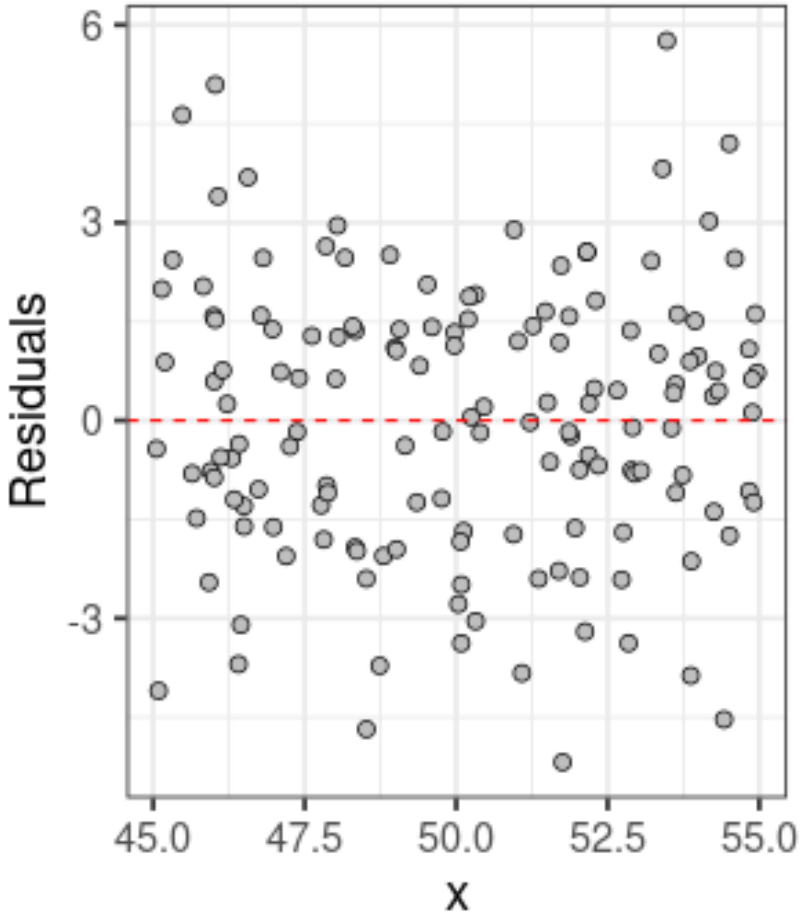
1. Plot histogram of residuals (normality)
2. Plot residuals against covariate (linearity, constant variance)
3. Plot residuals against new covariates (pattern identification)



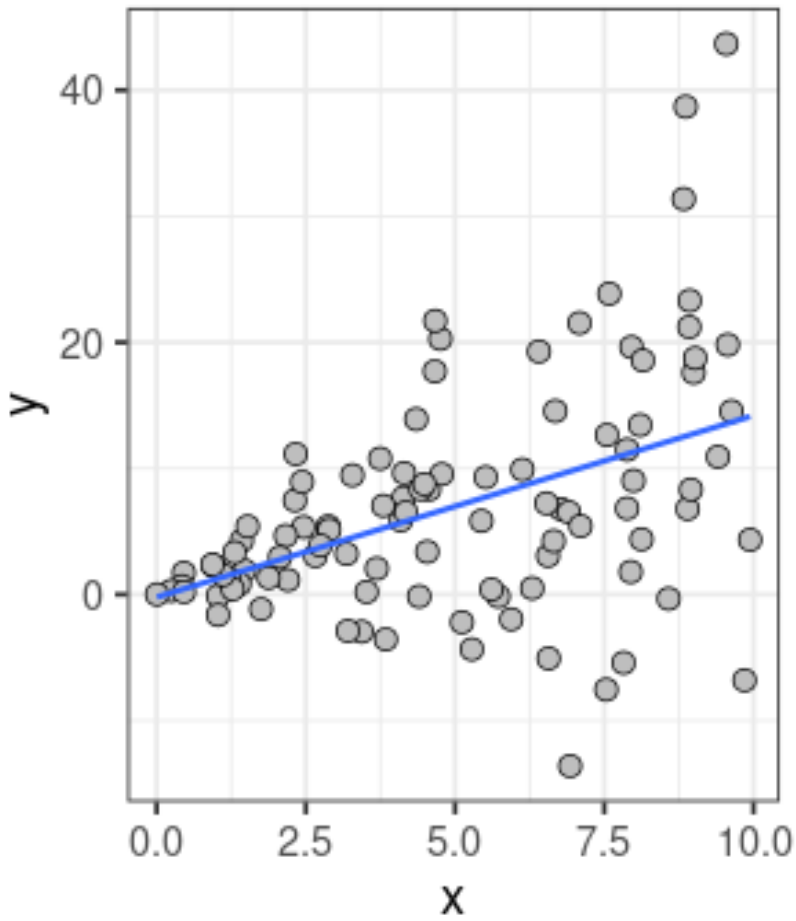
Are my errors normal?

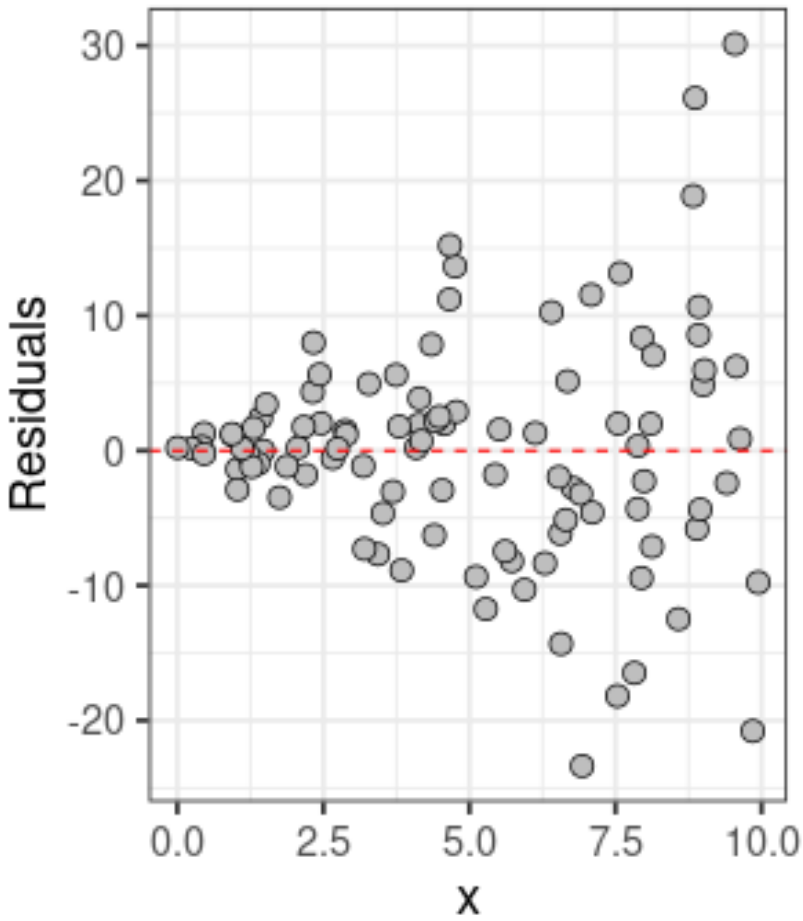
Is the variance the same at all levels of $X$?



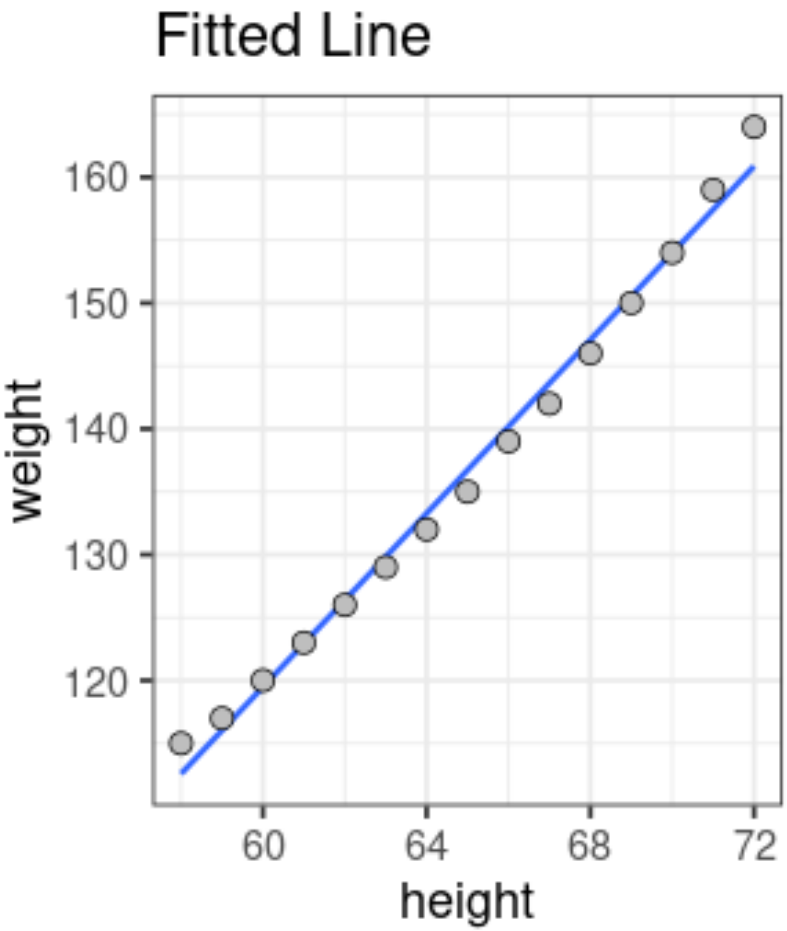



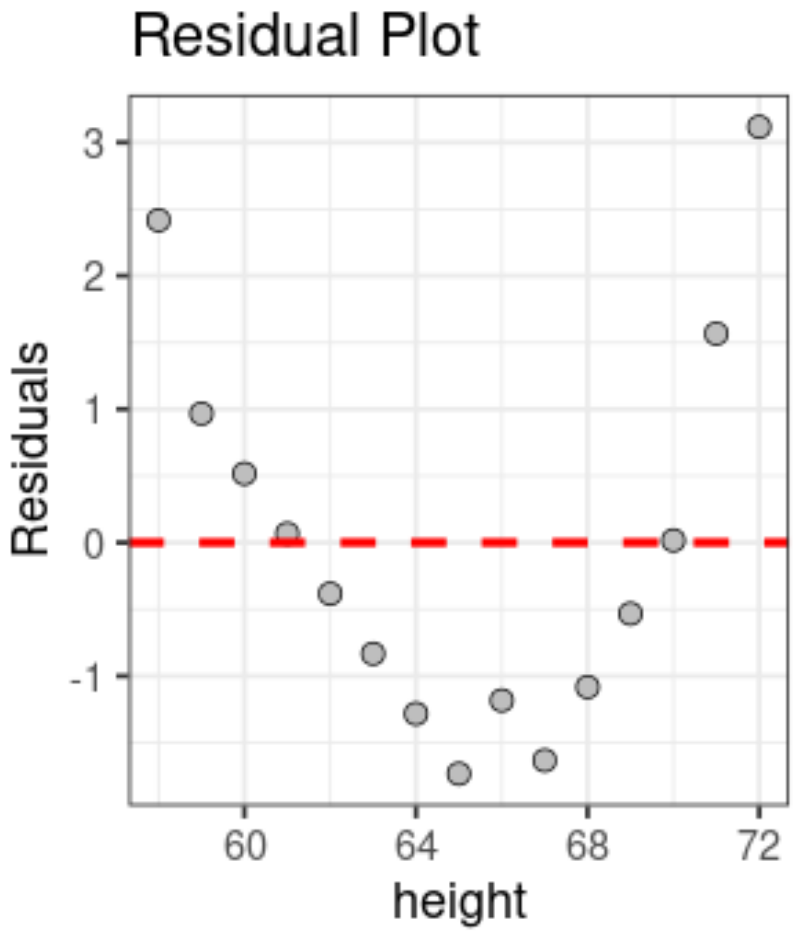
# Constant Variance



# Tests of linearity



# Part 2: Investigating Patterns

# Correlated Covariates

Consider a simple linear model in which a covariate $X$ is used to predict some value $y$

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$

The residuals associated with this describe the amount of variability that *is yet to be explained*

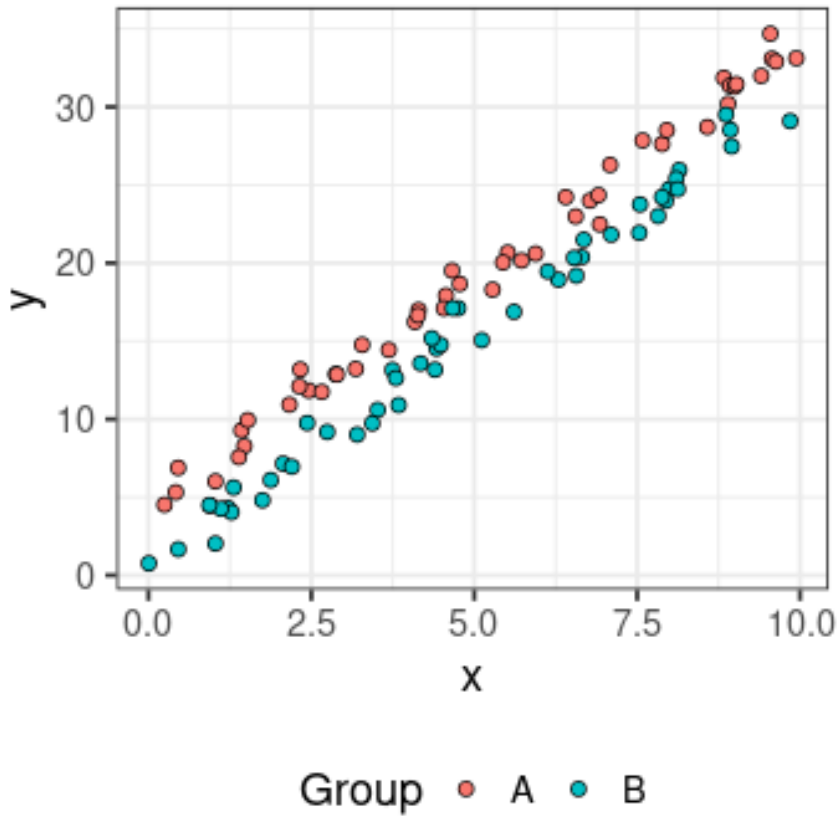
$$r = \hat{y} - y$$

The idea is to find new covariates *associated* with this residual, in effect "mopping up" the remaining uncertainty

# Considering new covariates (categorical)

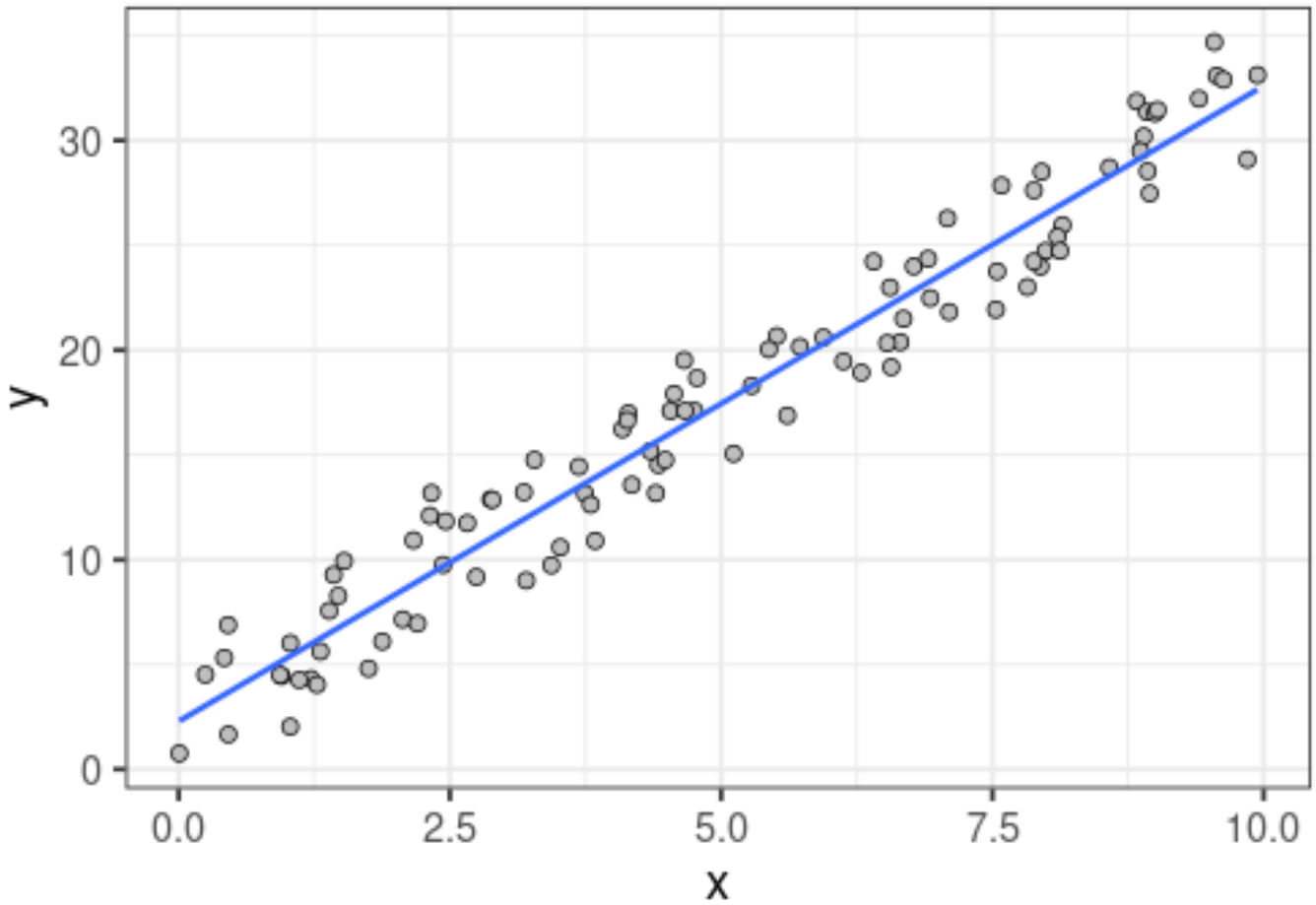
Suppose I have:

- Quantitative outcome $y$
- Quantitative predictor $X$
- Categorical predictor indicating group



# Considering new covariates (categorical)

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$



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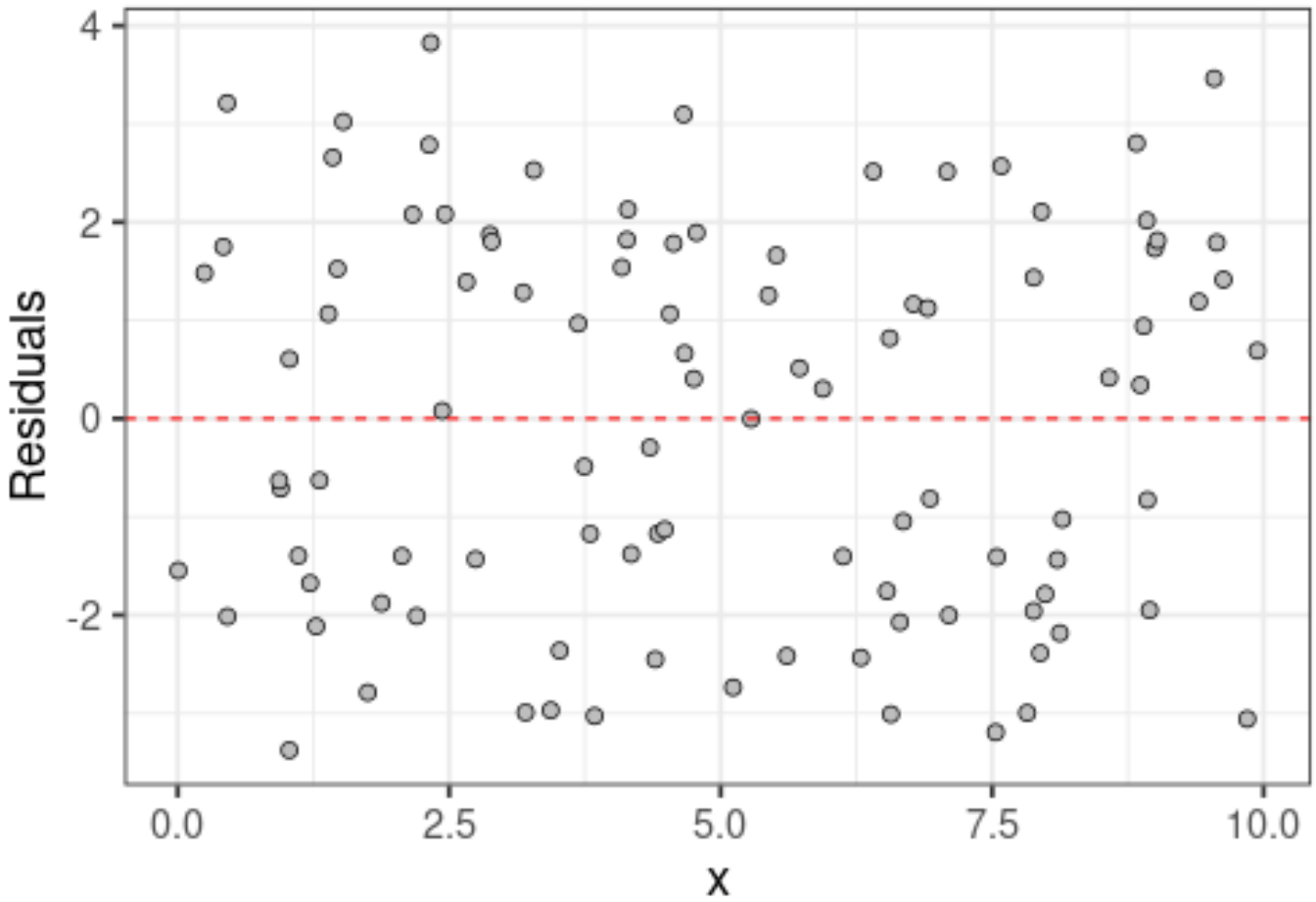

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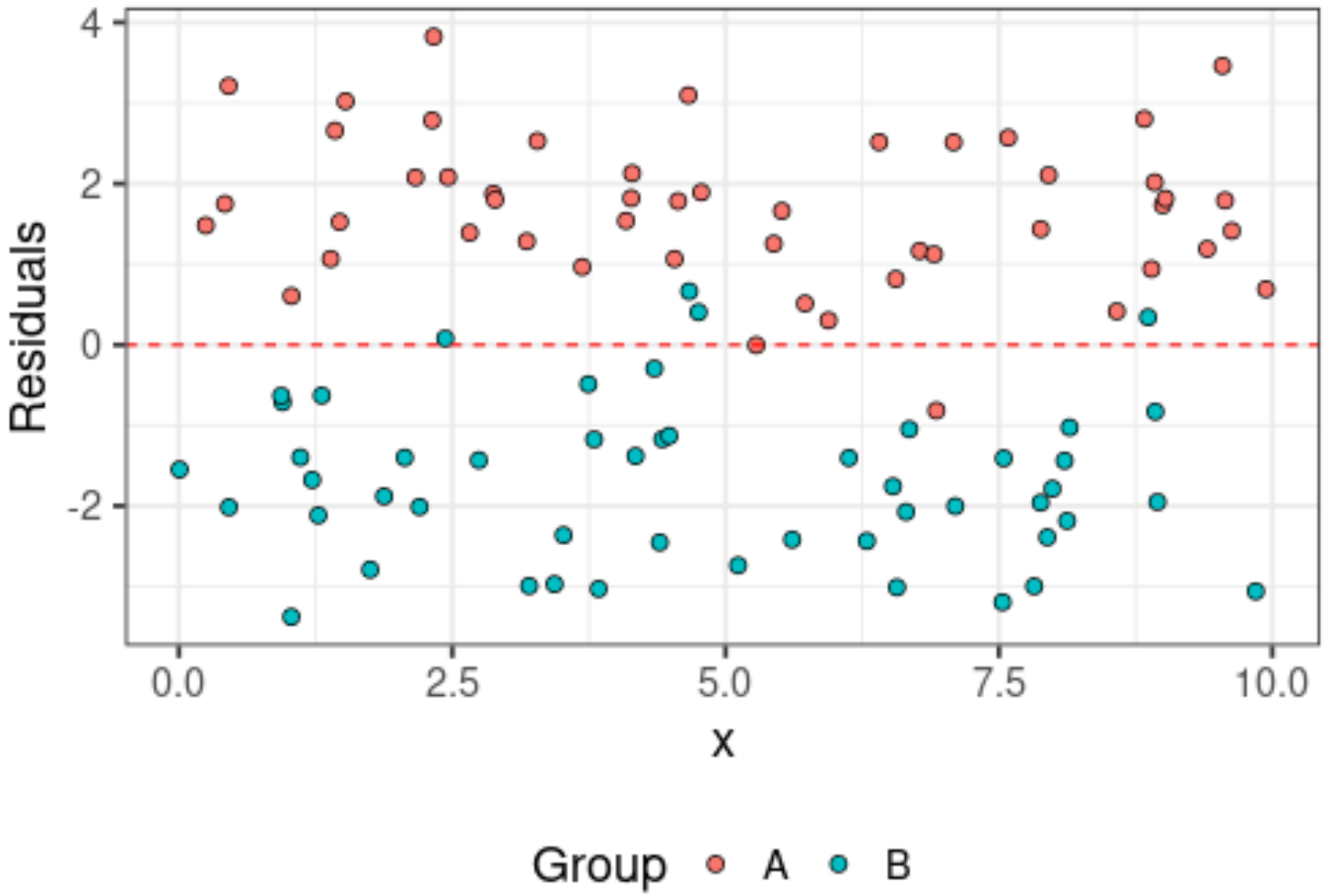
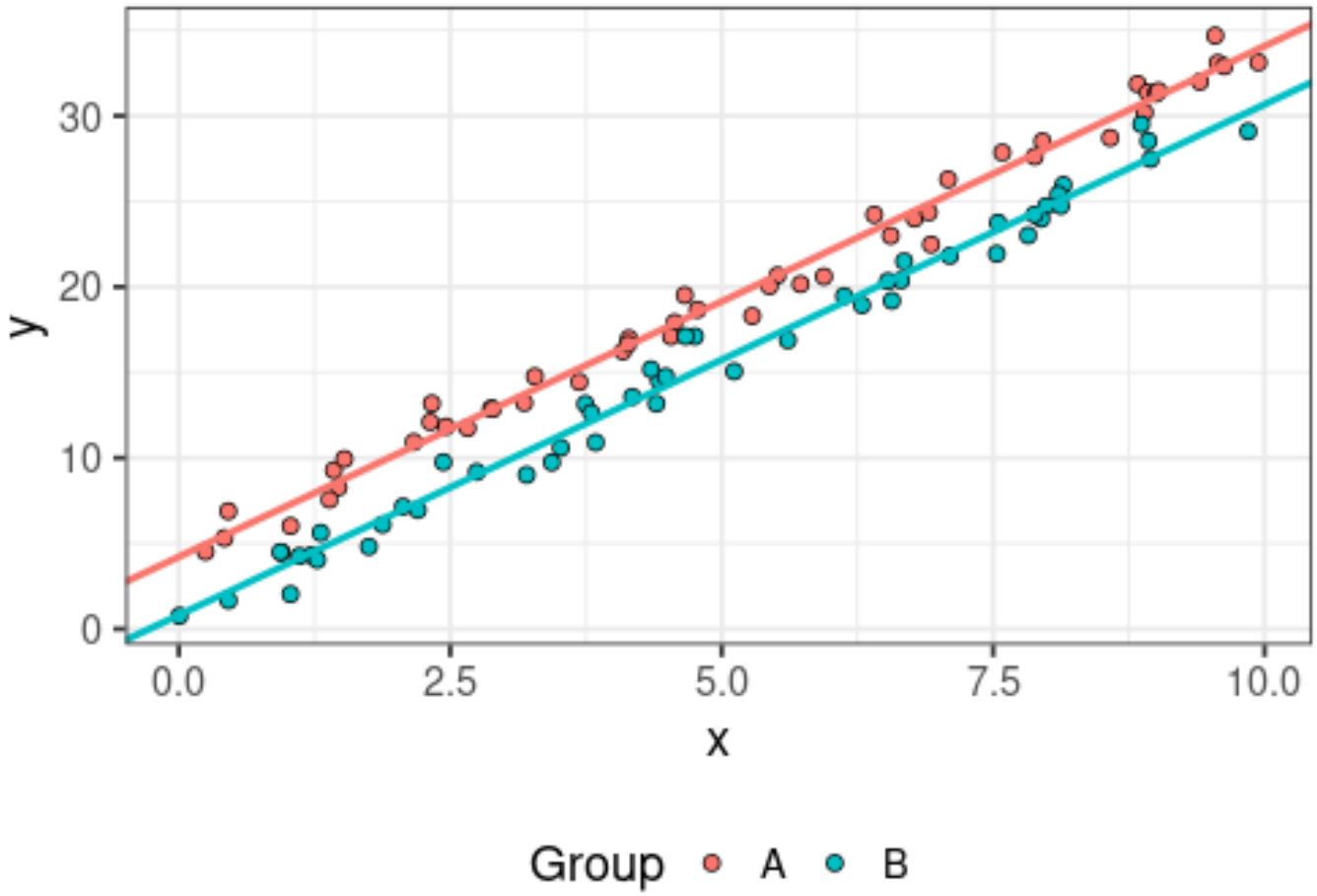
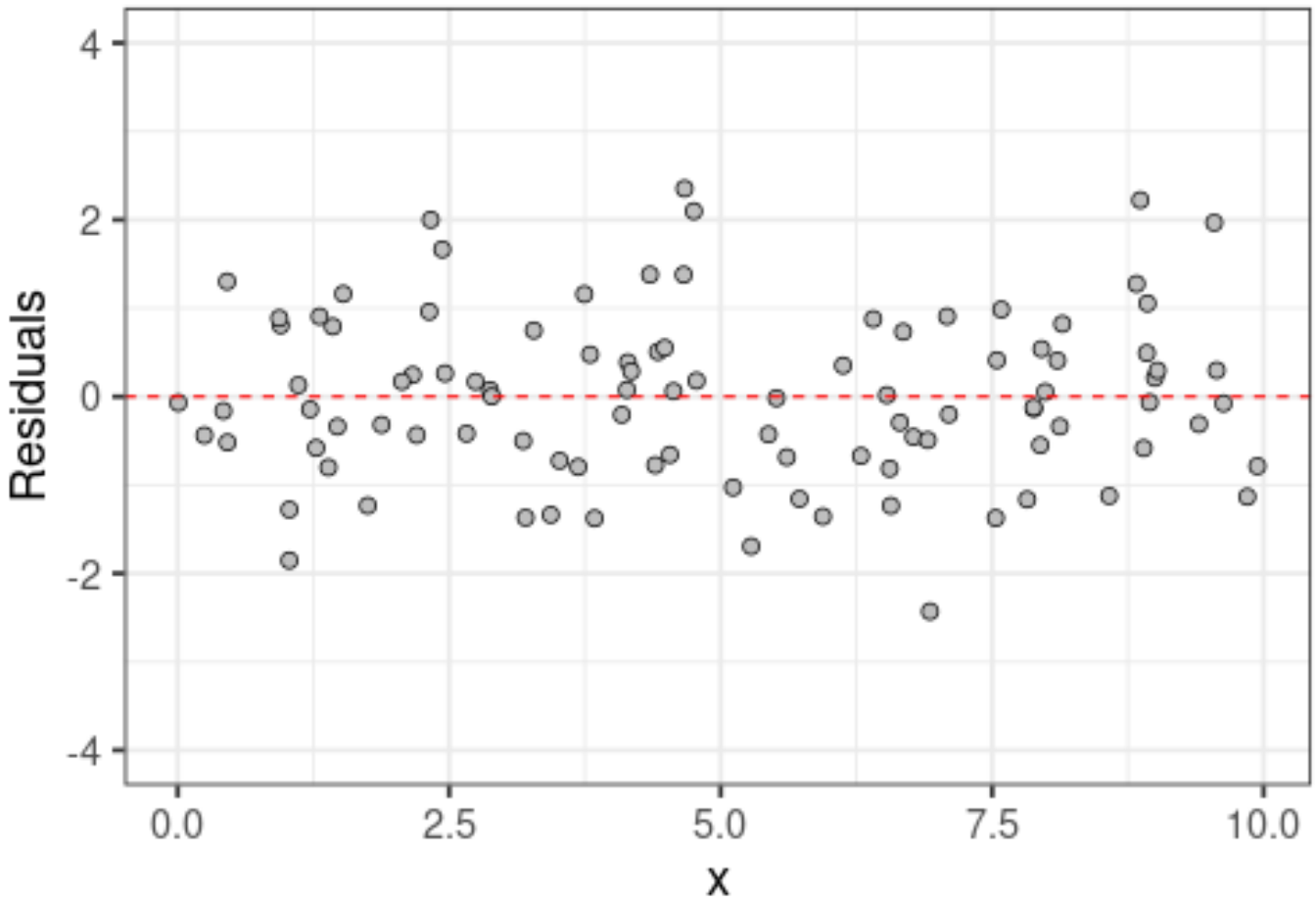


# Considering new covariates (categorical)

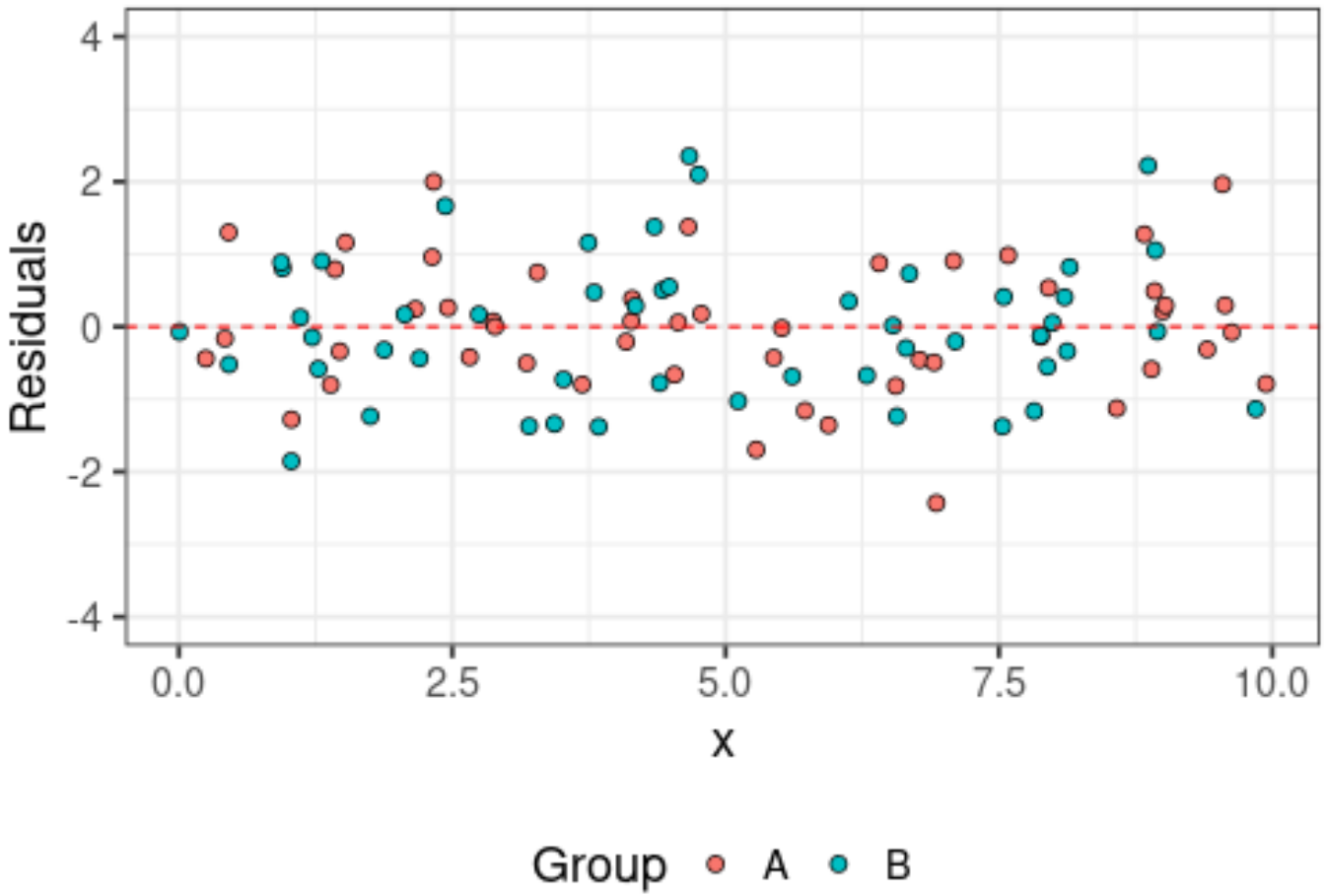
$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1 + \mathbb{1}_A\hat{\beta}_2$$



# Considering new covariates (categorical)



# Considering new covariates



# $R^2$ and Significance of Variables

```
> lm(y ~ x, df) %>% summary()

            Estimate Std. Error t value      Pr(>|t|)
(Intercept)   3.5283     0.1989    17.7 <0.0000000002 ***
x             3.0187     0.0347    86.9 <0.0000000002 ***
R-squared = 0.987

> lm(y ~ x + Group, df) %>% summary()

            Estimate Std. Error t value       Pr(>|t|)
(Intercept)   3.8999     0.2087   18.68 < 0.0000000002 ***
x             3.0186     0.0325   93.00 < 0.0000000002 ***
GroupB       -0.7419     0.1898   -3.91        0.00017 ***
R-squared = 0.989
```

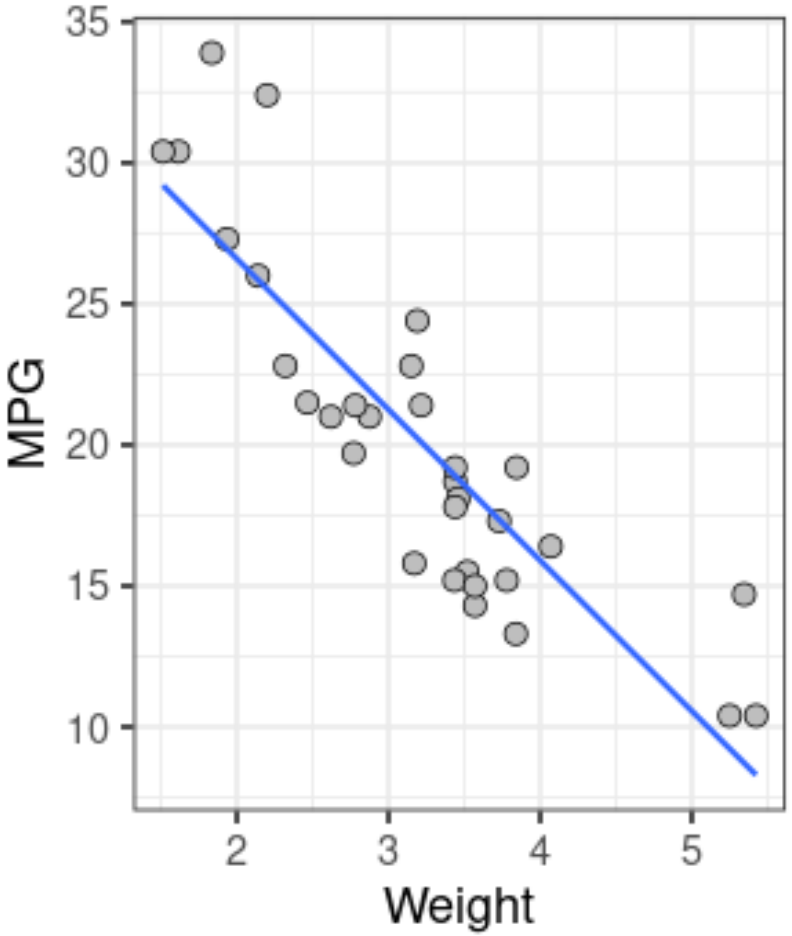
# Considering new covariates (quantiative)

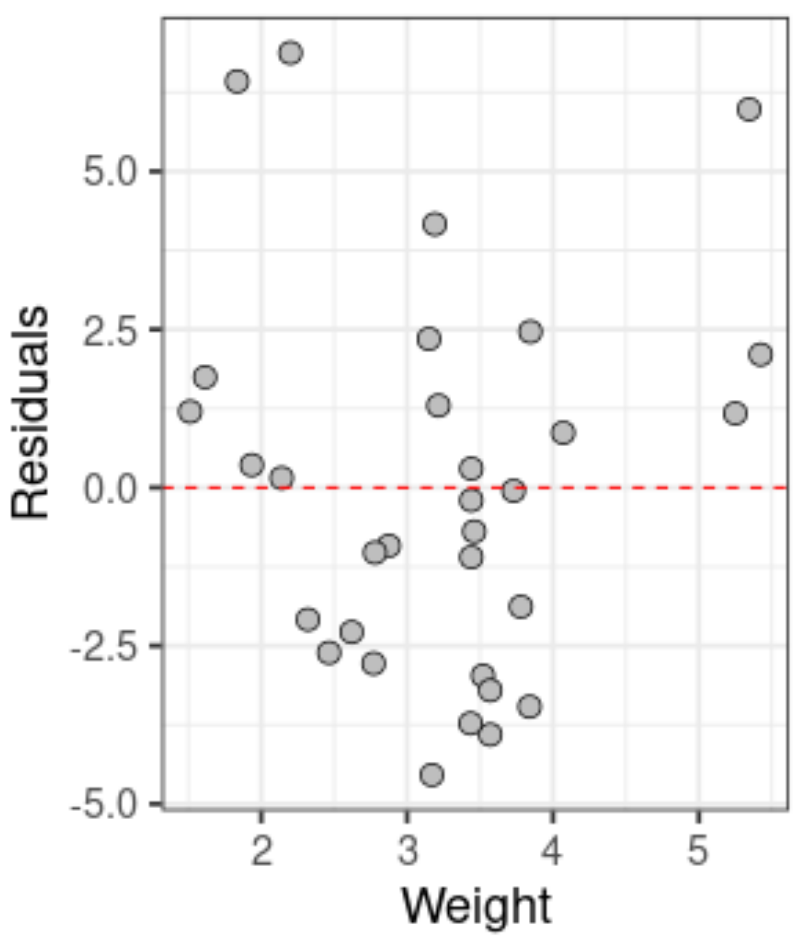
Now consider a situation in which we wish to predict fuel economy with three separate models:

1. Using weight
2. Using weight and engine displacement
3. Using weight and quarter mile time

Starting with the first model, we can consider the relationship of the residuals with both engine displacement and quarter mile time

# Considering new covariates (quantiative)

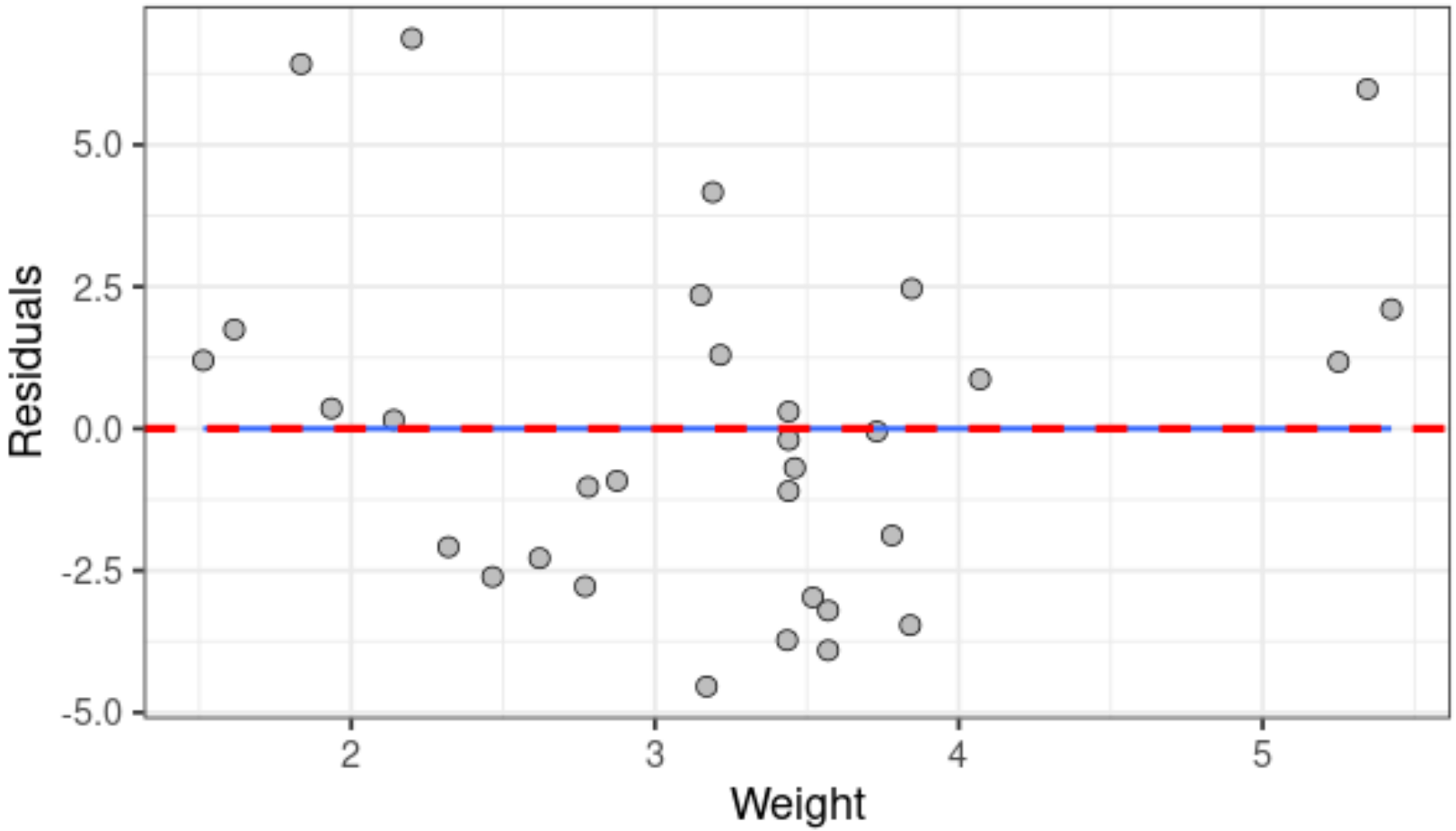


# Considering new covariates (quantiative)

An interesting thing occurs when we try to create a regression model of the residuals with the original variable:

$$\text{residuals} = \hat{\beta}_0 + \hat{\beta}_1 \times \text{Weight} = 0$$



# Considering new covariates (quantiative)

When considering adding new variables to our regression model, we want to add those that will "mop up" the residuals that are left after considering weight

This brings us to the idea of **correlated variables**, or variables that have evidence of a *linear relationship* with one another

How correlated our variables may be will impact how much of the residual they are able to account for

# Correlated Covariates

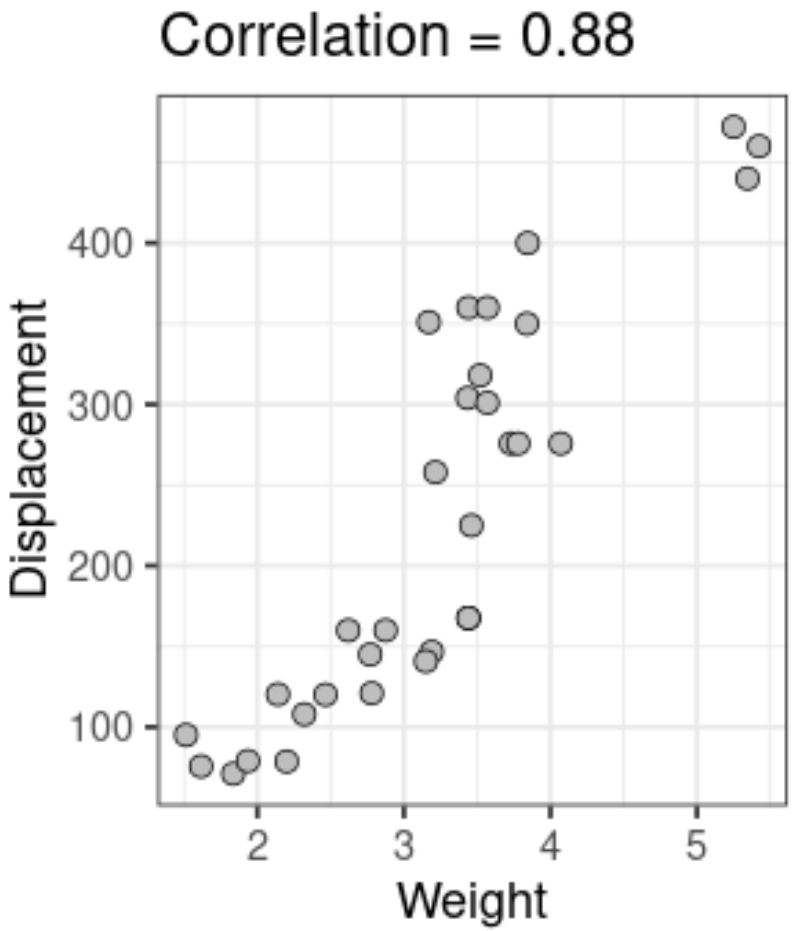
We can consider two extremes: if two quantitative variables are perfectly correlated, knowing the value of one variable means we also know the value of the other

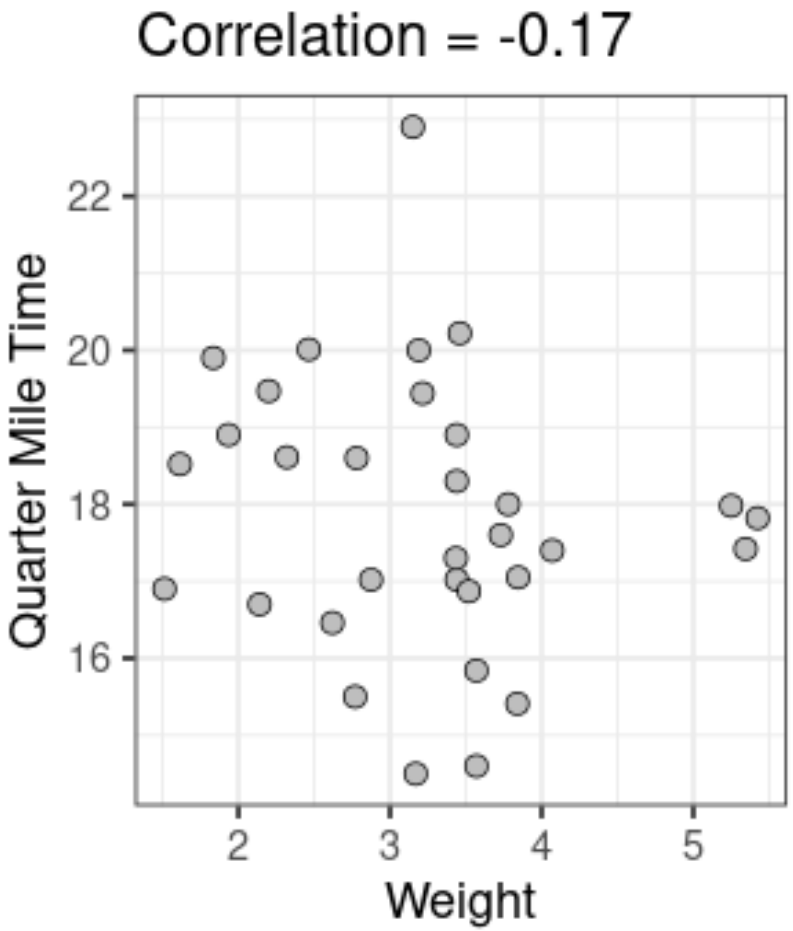
This means that, in terms of predicting an outcome, adding a highly correlated variable to our model will contribute little new information and will not be very useful

By constrast, if two variables have perfectly uncorrelated, then knowing the value of one tells us nothing about the value of another

Adding an uncorrelated variable to our model thus offers more potential to "mop up" the variability that was not explained by the first variable

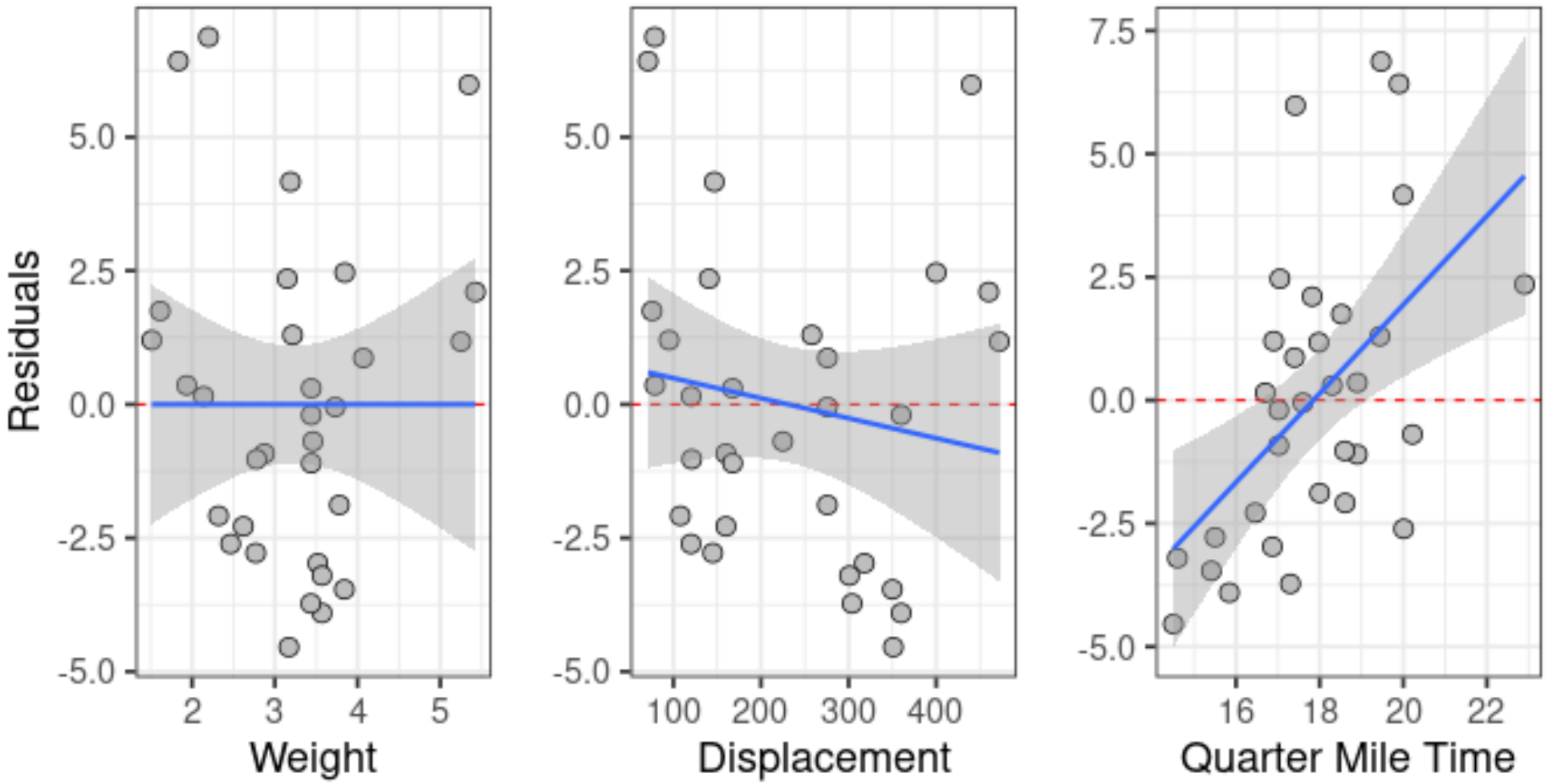
# Correlated Covariates



# Residual Plots

Displacement has little association with the residuals, while quarter mile time has quite a bit of association. This suggests that adding QM time will "mop up" more unexplained variance



# Correlated Covariates

```
> lm(mpg ~ wt, mtcars) %>% summary()

            Estimate Std. Error t value     Pr(>|t|)
(Intercept)   37.285      1.878   19.86   < 0.000002 ***
wt            -5.344      0.559   -9.56     0.000013 ***
R-squared = 0.75    Adj R-squared = 0.745

> lm(mpg ~ wt + disp, mtcars) %>% summary()

            Estimate Std. Error t value      Pr(>|t|)
(Intercept) 34.96055    2.16454   16.15  0.000000049 ***
wt          -3.35083    1.16413   -2.8         0.0074 **
disp        -0.01772    0.00919   -1.93        0.0636 .
R-squared = 0.78    Adj R-squared = 0.766

> lm(mpg ~ wt + qsec, mtcars) %>% summary()

            Estimate Std. Error t value       Pr(>|t|)
(Intercept)   19.746      5.252    3.76        0.00077 ***
wt            -5.048      0.484  -10.43 0.000000000025 ***
qsec           0.929      0.265    3.51        0.00150 **
R-squared = 0.82    Adj R-squared = 0.814
```

# Key Takeaways

1. Number of assumptions for linear model
   - Linearity
   - Normal errors
   - Constant Variance
2. Need way to determine which new variables to add to model
3. Examining errors effective way to test assumptions and investigate new covariates
4. Relationship between correlation of predictors and residual analysis