Confidence Intervals II

Grinnell College

March 23, 2026

Warm-up

1. What is a sampling distribution? How does this relate to CLT?
2. What determines uncertainty in my estimate of $\overline{X}$?
3. What is a confidence interval? How is it constructed?
4. How does standardizing $\overline{X}$ help us find critical values?

Review

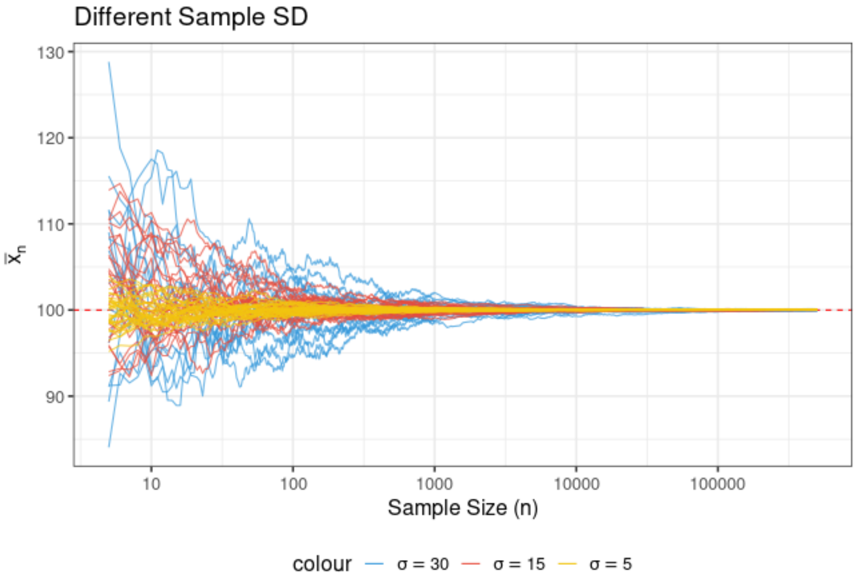
The **Law of Large Numbers** guarantees that, as the number of observations n in my sample increases, my estimate of the parameter will converge to the true value

A **sampling distribution** refers to the distribution of a sample statistic (i.e., $\overline{x}$) if we were to repeatedly sample from a population and recompute the statistic

- What values would they take?
- How frequently would they appear?

The **Central Limit Theorem** states that is my statistic is an average or a proportion, then the sampling distribution of my statistic will be approximately normal, with

$$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$


Different Sample SD
130
120
110
100
90
$\bar{x}_n$
10
100
1000
10000
100000
Sample Size (n)
colour
σ = 30
σ = 15
σ = 5

Standardization

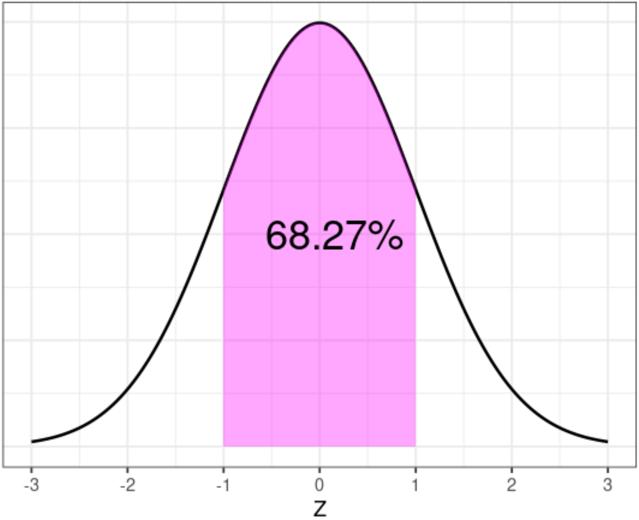
Recall that we can standardize our sample mean creating a Z score, where

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

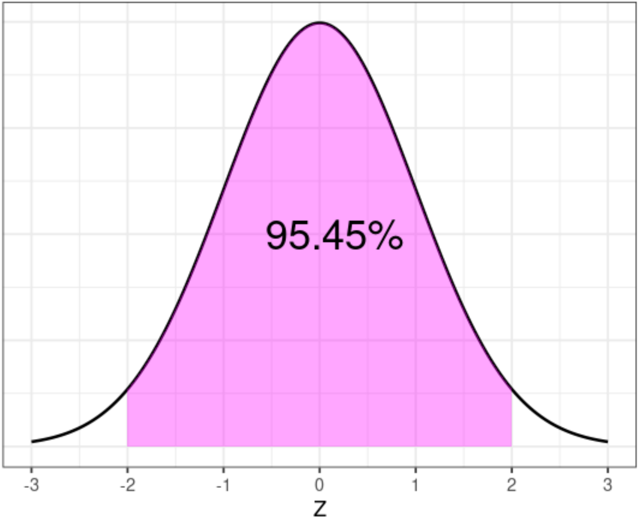
To construct a confidence interval with $M\%$ confidence, we simply need to find C such that

$$P(-C < Z < C) = M\%$$

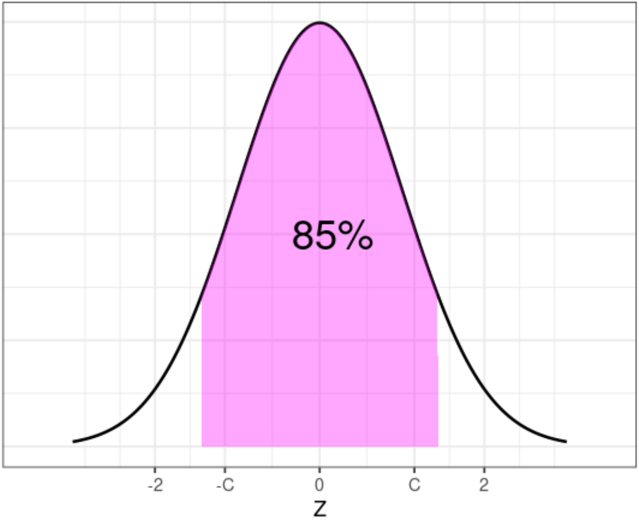
$$P(-1 < Z < 1) = 0.68$$



$$P(-2 < Z < 2) = 0.95$$



$$P(-C < Z < C) = 0.85$$



Point $\pm$ Margin of Error

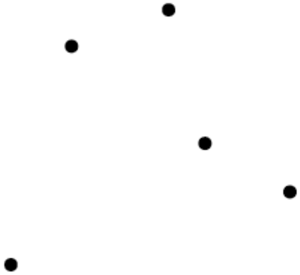
The key idea is this:

In finding a plausible range for our parameter μ, we want to create an interval, centered at our observed statistic, that takes the form

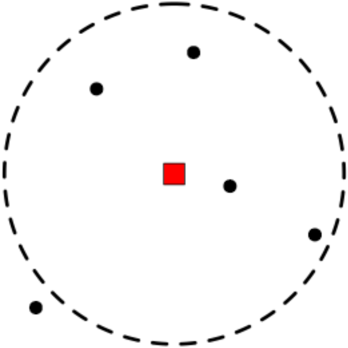
$$\overline{X} \pm C \times \left(\frac{\hat{\sigma}}{\sqrt{n}} \right)$$

Our **confidence**, mediated by C, is the frequency with which an interval constructed in this way will contain the true value of the mean.

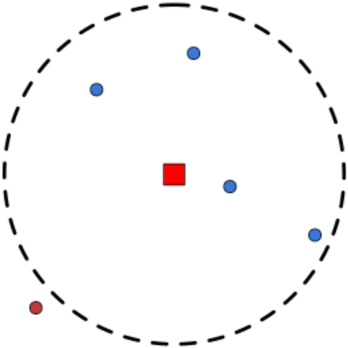
Start with a collection of different sample means, $\overline{X}_1, \overline{X}_2, \ldots$



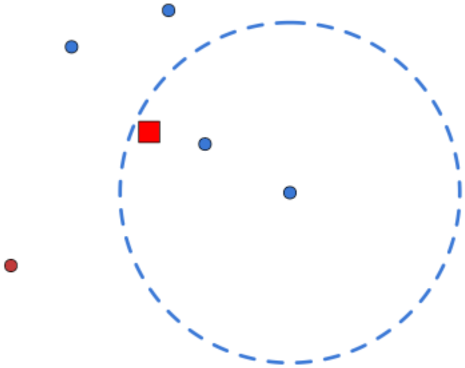
The CLT tells us how the sample means will be distributed:
$\overline{X} \sim N(\mu, \sigma/\sqrt{n})$



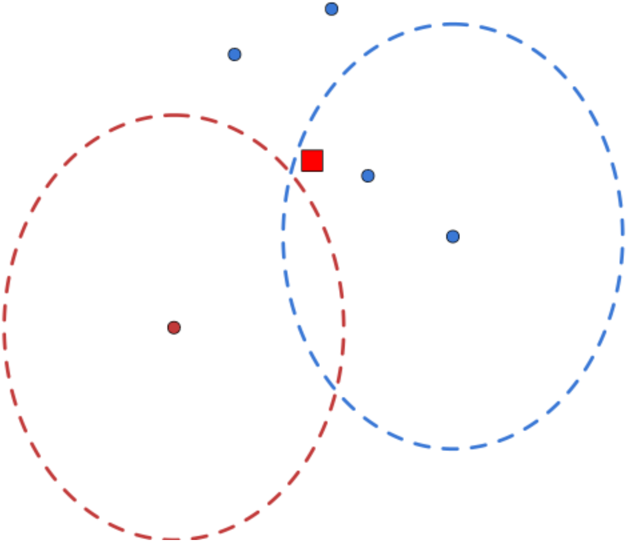
Depending on our choice of critical value, C, some percentage of sample means will fall within some interval of μ

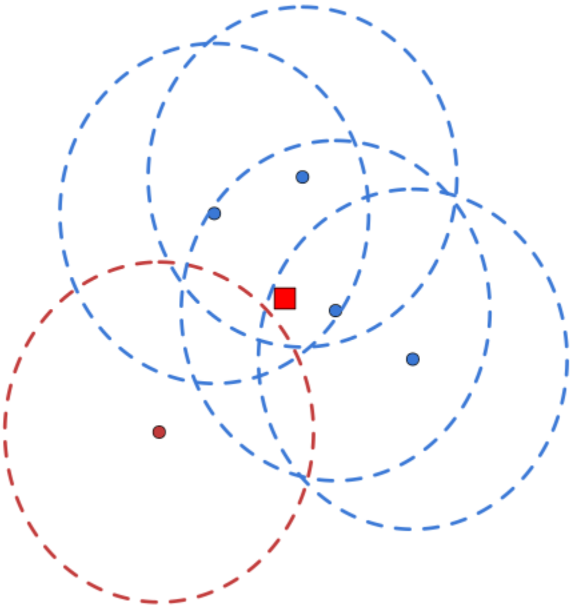


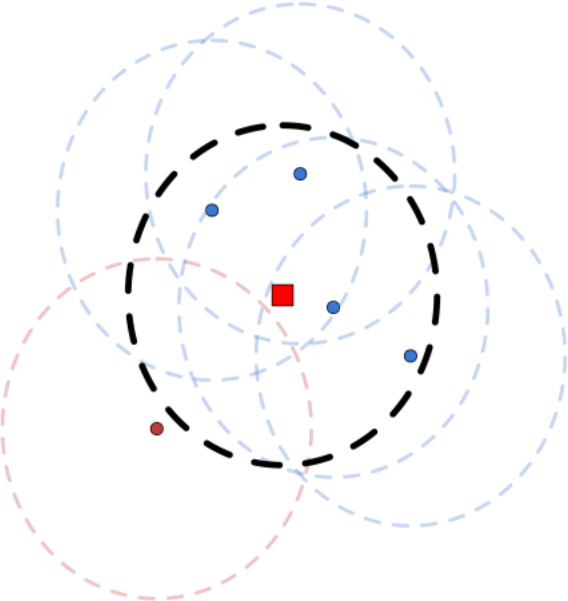
By *symmetry*, if some $\overline{X}$ falls in an interval around μ, μ will fall within an interval around that $\overline{X}$

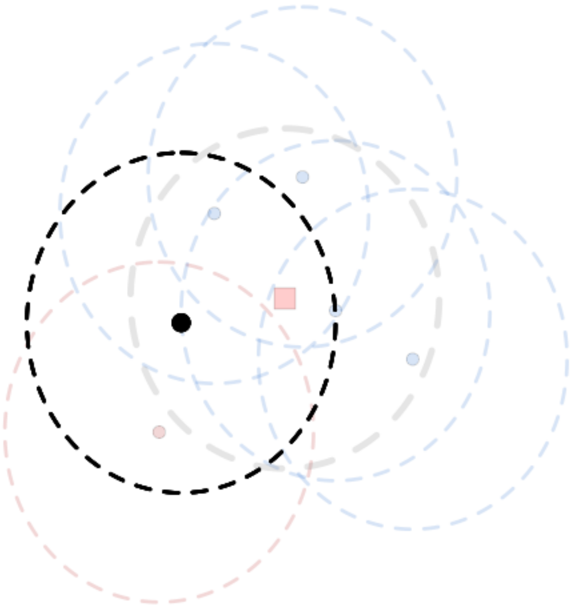


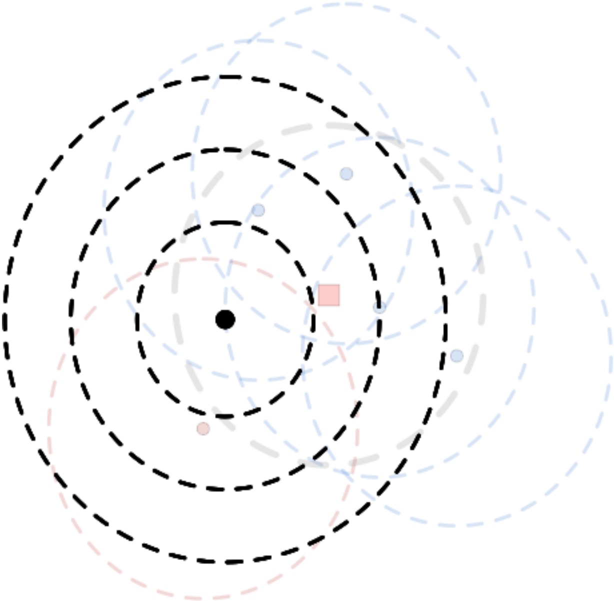
This is also true if a particular $\overline{X}$ does *not* fall within an interval around μ











Key Ideas Today

We need to have clear two separate but not independent ideas in our minds: the size of a an interval and the amount of confidence

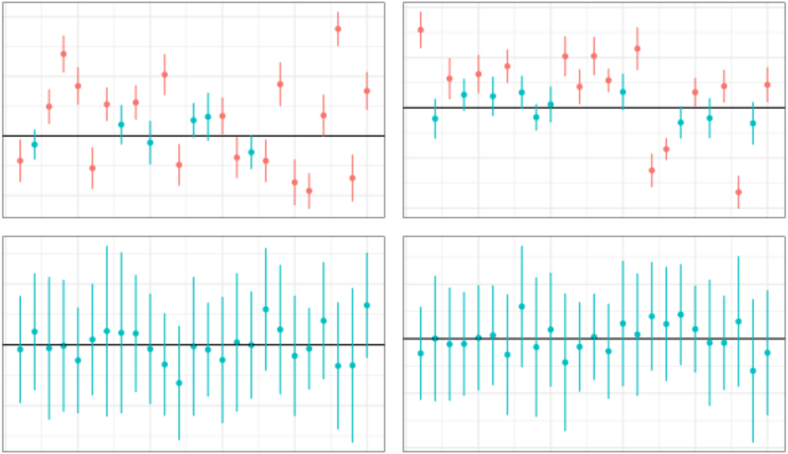
CRITICAL VALUES ARE THE ONLY THING THAT DETERMINE CONFIDENCE $\leftarrow$ This is a fact, never let anyone tell you otherwise

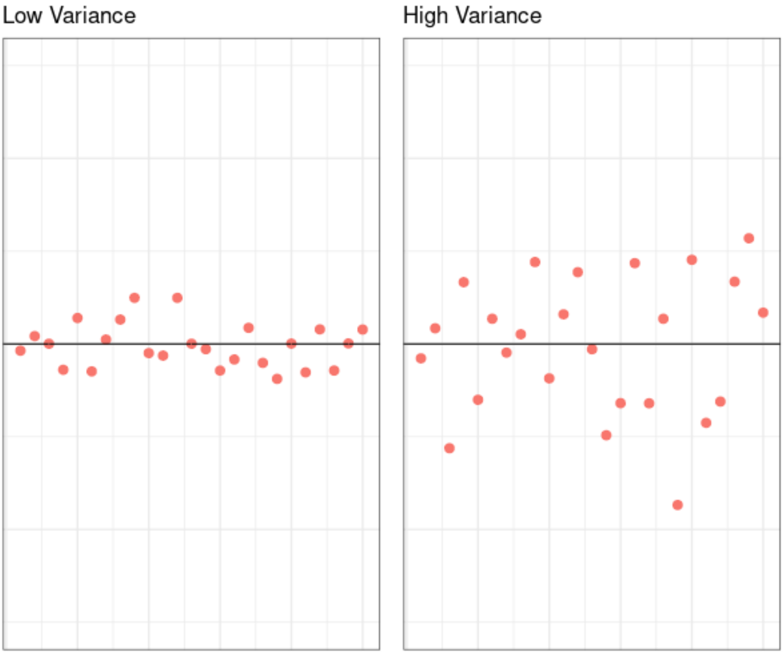
While the critical values are the only thing that determine confidence, there are in fact three pieces that determine the final size:

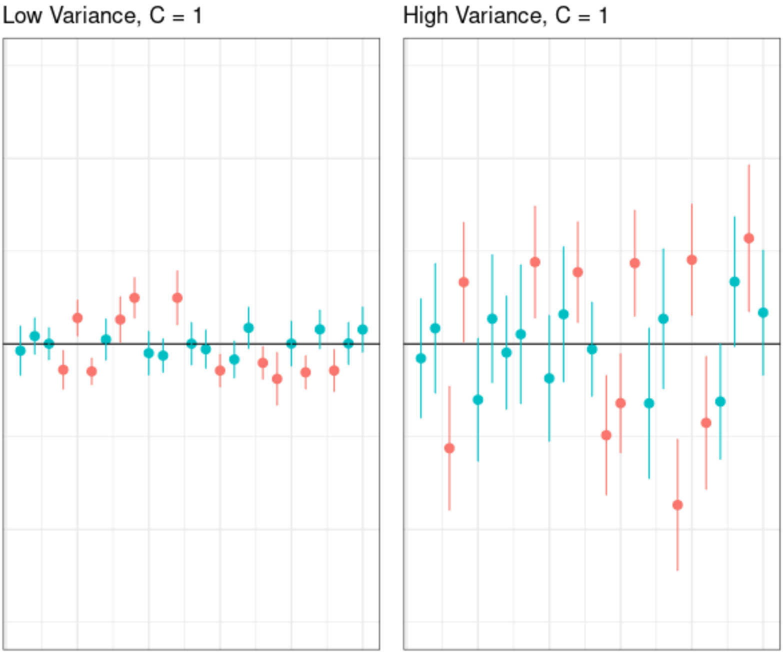
- Critical value, C (more confidence means larger intervals)
- The standard error ($\hat{\sigma}/\sqrt{n}$), communicating the amount of uncertainty in $\overline{X}$ (more uncertainty also means larger intervals)

Confidence Intervals

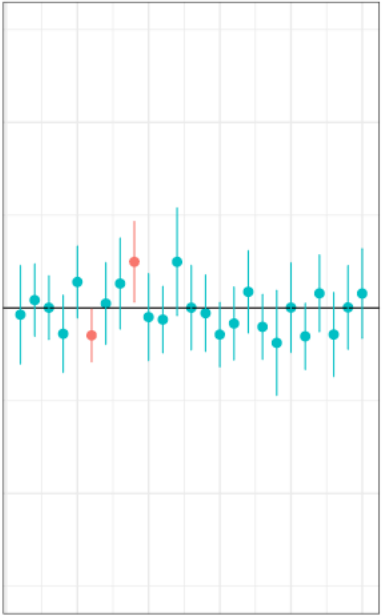
It is also worth observing that we can *alter* our process to achieve different results. There is a trade-off between how frequently we are correct and how much uncertainty we allow in our prediction



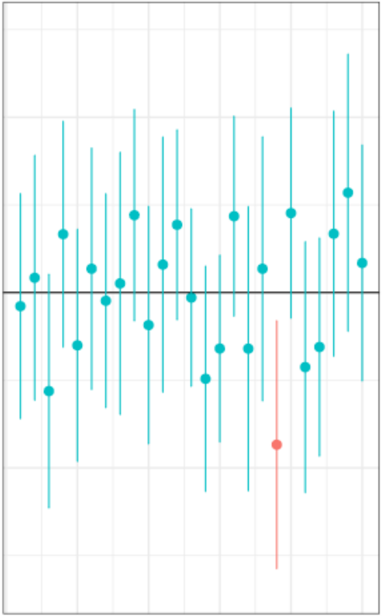

Low Variance
High Variance

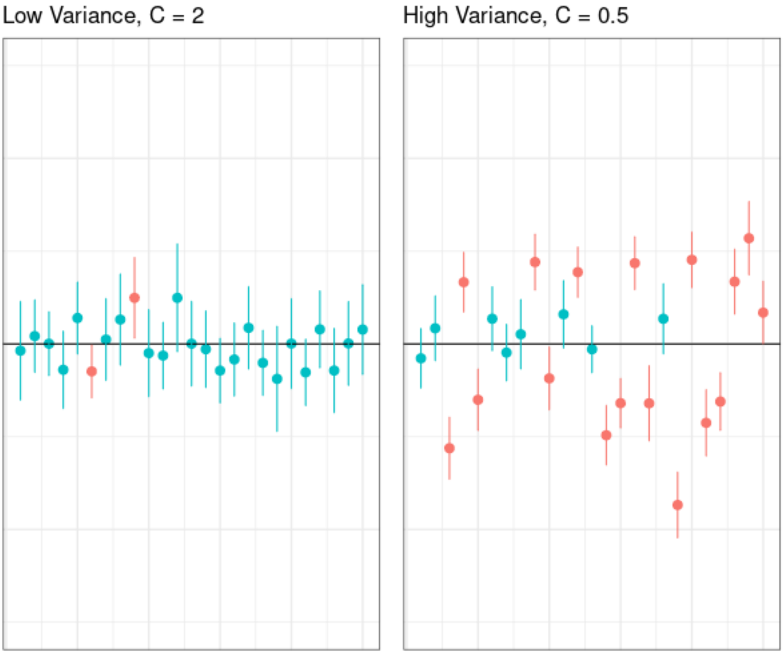

Low Variance, C = 1
High Variance, C = 1

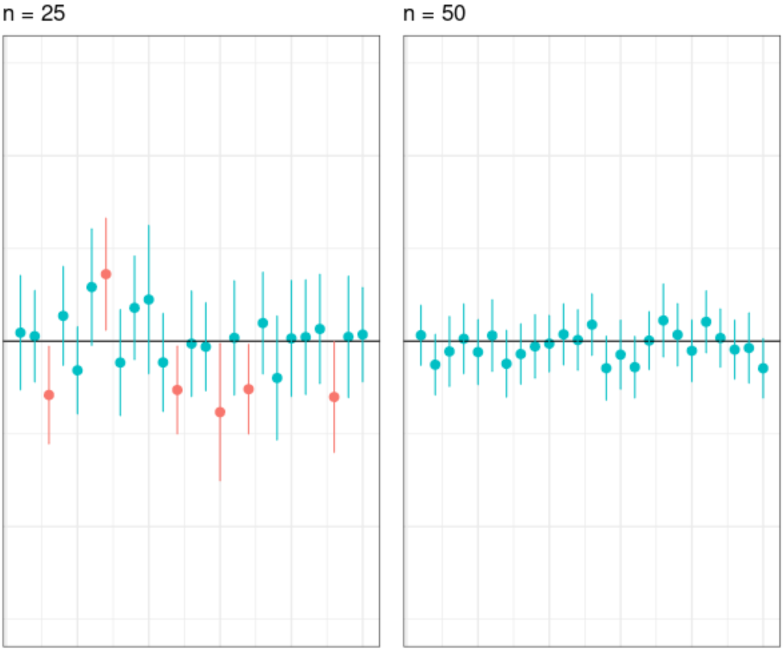
Low Variance, C = 2



High Variance, C = 2




Low Variance, C = 2
High Variance, C = 0.5


n = 25
n = 50

Scenarios

What kind of behavior should you expect in these scenarios:

Scenario 1: Small n, large $\hat{\sigma}$, small C

Scenario 2: Large n, large $\hat{\sigma}$, large C

Scenario 3: Small n, small $\hat{\sigma}$, large C

Identify: what are sources of variability in $\overline{X}$, what determines my confidence

Review

What are we doing:

1. Collecting a sample to estimate plausible interval for population parameter
2. Based on mean and standard deviation from sample
3. Interval constructed according to CLT

What impacts my intervals:

1. Critical value C *only thing that mediates confidence*. Larger C means larger interval (why)
2. More variability in $\overline{X}$ also means larger interval. The variability in $\overline{X}$ is measured by the *standard error*, $\hat{\sigma}/\sqrt{n}$
   - $\hat{\sigma}$ is the estimate of variability in the population. If this is larger, uncertainty in $\overline{X}$ will be large (and vice versa)
   - n represents the sample size. Larger n means more data means more precise estimate of $\overline{X}$