Bayes' Theorem

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Addition Rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Multiplication Rule:

$$\begin{aligned} P(A \text{ and } B) &= P(B|A) \times P(A) \\ &= P(A|B) \times P(B) \end{aligned}$$

Conditional Probability Rule (from Multiplication):

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Goals

Our goal today is to learn Bayes' Theorem:

- What does it say?
- Why is it true?
- When would we use this?

Bayes' Theorem is a statement that allows us to invert a conditional probability with the following relation:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

In particular, Bayes' Theorem gives us a way to *update* our beliefs about an event (say, the probability of A) in light of new information being provided (i.e., event B)

It also allows us to use prior information with respect to rare outcomes to provide context for data that is observed (i.e., the *base-rate fallacy*)

Example

Suppose we took a collection of 9 individuals, giving 5 of them a placebo and 4 of them an active drug. We then recorded how many from each group were sick once in the next 100 days

	Sick	Not Sick	Total
Placebo	3	2	5
Drug	1	3	4
Total	4	5	9

Sum of Conditional Probabilities

For any event, B, the sum of conditional probabilties will *always* be equal to 1. Here, we condition on an individual being given a placebo

	Sick	Not Sick	Total
Placebo	0.6	0.4	1
Drug	0.25	0.75	1

$$1 = P(\text{Sick}|\text{Placebo}) + P(\text{Not Sick}|\text{Placebo})$$

In general, for a set of *disjoint* and *exhaustive* events $A_1, \dots, A_k$ such that

$$P(A_1) + P(A_2) + \dots P(A_k) = 1$$

it will follow that

$$P(A_1|B) + P(A_2|B) + \dots P(A_k|B) = 1$$

Because we can break every space into two sets, A and A^C, we can simplify with the following relation

$$1 = P(A|B) + P(A^C|B)$$

or

$$P(A|B) = 1 - P(A^C|B)$$

The utility in knowing that

$$1 = P(A|B) + P(A^C|B)$$

is that it allows us to state the following relation:

$$P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$$

Practice 1

Suppose a drug test for cannabis is 90% sensitive, meaning that the probability that it comes back positive for a cannabis user is 90%. Further, the true negative rate is 80%, meaning that the test will be negative for a non-cannabis user 80% of the time

Assume that 5% of a population uses cannabis, what is the probability that somebody who tests positive is actually a cannabis user?

Practice 2

In Canada, about 0.35% of women over 40 will develop breast cancer in a given year. A mammogram is a low-cost, non-invasive procedure for testing for breast cancer, but it is not perfect. In about 11% of patients with breast cancer, it will return a **false negative**. Similarly, the test will give a **false positive** in 7% of patients who do not have breast cancer.

If a random woman over 40 is tested for breast cancer using a mammogram and the test comes back positive, what is the probability that the patient actually has breast cancer?

Generalizability

As a brief note, Bayes' Theorem does not require that we only have two cases. If $A_1, A_2, \ldots, A_k$ represent all of the possible disjoint outcomes in a sample space, Bayes' Theorem can be written

$$\begin{aligned} P(A_1|B) &= \frac{P(B|A_1)P(A_1)}{P(B)} \\ &= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots P(B|A_k)P(A_k)} \end{aligned}$$

OpenIntro Statistics, 4th Edition