Probability (Part 1)

Grinnell College

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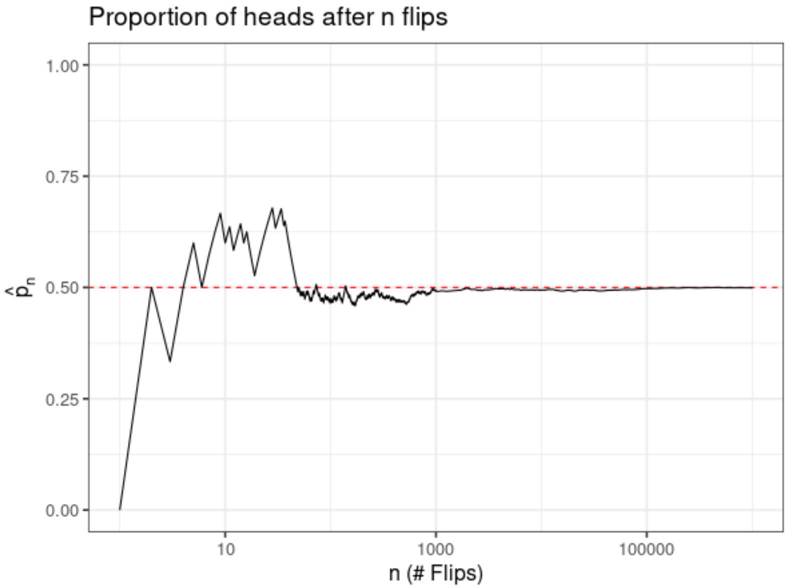
We will be concerning ourselves with *outcomes* associated with *random processes*

Probability of an *outcome* is the proportion of times the outcome would occur if we observed the *random process* an infinite number of times

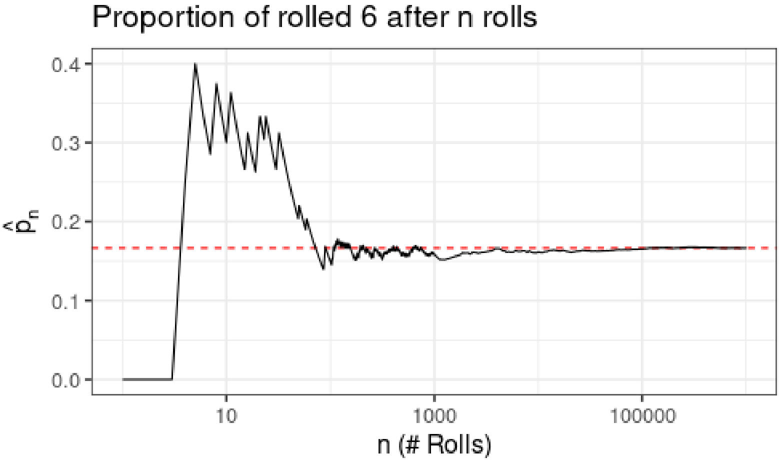
Simple examples include:

- Flipping a coin
- Rolling a dice
- Sampling marbles from a jar
- Drawing a card from a deck



Proportion of heads after n flips
1.00
0.75
0.50
0.25
0.00
$\hat{p}_n$
10
1000
100000
n (# Flips)

As more observations are collected (n increases), the size of fluctuations of p_n around p will begin to shrink. This tendency to stabilize is known as the **Law of Large Numbers**



A set of all possible outcomes, denoted $\mathcal{S}$, is called a **sample space**.

Consider rolling a dice, where the set of possible outcomes is

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

We express probability of an outcome as such

$$P(\text{rolling a 6}) = \frac{1}{6}$$

If context is clear, we can make it simpler:

$$P(6) = \frac{1}{6}$$

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

Two outcomes are said to be **disjoint** or **mutually exclusive** if they cannot both happen at the same time. When two outcomes are disjoint, finding their probability follows a simple rule:

$$P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

We call this the **Addition Rule**

Notes on Notation

$$A = \{2, 3, 4\} \qquad B = \{3, 4, 5\}$$

Then

- A or $B = \{2, 3, 4, 5\}$
- A and $B = \{3, 4\}$

In set theory terms, these are known as *unions* and *intersections*

- A or $B \equiv A \cup B$
- A and $B \equiv A \cap B$

The **Addition Rule** states that if outcomes A_1 and A_2 are *disjoint*, then the probability of one of them occuring is

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

If there are many disjoint outcomes $A_1, \ldots, A_k$, then the probability that one of them will occur is

$$P(A_1) + P(A_2) + \cdots + P(A_k)$$

Are the following events disjoint?

- Using a dice to roll an even number or to roll a 3?
- Using a dice to roll an odd number or a number greater than 4?
- Drawing a diamond or drawing a face card from a standard deck of playing cards?

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

Suppose we specify two events:

- A: we roll a 1, 2, or 4
- B: we roll an odd number

How would we find the probability $P(A \text{ or } B)$?

The **General Addition Rule** states that for *any* events A and B, the probability that at least one will occur is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are *disjoint* events, what is $P(A \text{ and } B)$?

Practice

In a standard deck of 52 cards, there are 4 suits (diamonds, hearts, spades, and clubs), each containing 13 cards. Within each suit, there are 4 face cards (J, Q, K, and A)

What is the probability that we draw a card that is either a face card or a diamond?

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- What values?
- How frequent?

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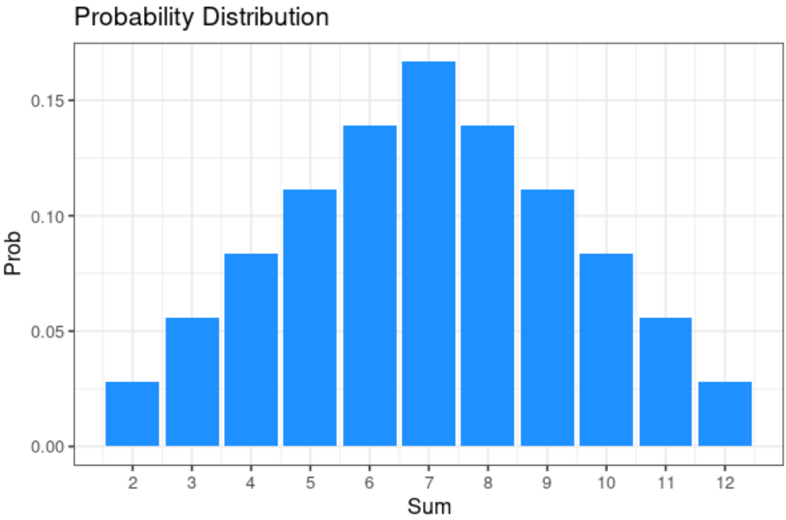
For a probability distribution to be valid, the following must be true:

1. The outcomes are disjoint
2. Every probability is between 0 and 1
3. The sum of all probabilities must equal 1

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



For an *event* A in our *sample space* $\mathcal{S}$, the **complement** of A, denoted A^C, represents all of the events in $\mathcal{S}$ that are not in A

Since A and A^C represent all possible events, it follows that $A \cup A^C = \mathcal{S}$

From the Addition Rule, we then have that

$$P(A \text{ or } A^C) = P(A) + P(A^C) = 1$$

Or, perhaps even more useful, we find that

$$P(A) = 1 - P(A^C)$$

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

If A represents *not* rolling a dice that sums to 8, we have

$$\begin{aligned} P(A) &= P(2) + \cdots + P(7) + P(9) + \cdots + P(12) \\ &= \frac{1}{36} + \cdots + \frac{6}{36} + \frac{4}{36} + \cdots + \frac{1}{36} \\ &= 31/36 \end{aligned}$$

However, if A^C is the event that we *do* roll an 8, we could more easily find

$$P(A) = 1 - P(A^C) = 1 - \frac{5}{36} = \frac{31}{36}$$

Key Terms

- **Probability** of an outcome is proportion of times outcome would occur if repeated infinite number of times
- **Law of Large Numbers** is tendency for empirical proportion of events to converge to probability
- Events are **disjoint** or **mutually exclusive** if they cannot both happen at the same time
- The **Addition Rule** states that $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- A **probability distribution** represents the probabilities of all disjoint outcomes of a random process
  - What values?
  - How frequent?
- The **complement** of an event A, denoted A^C, is the set of all possible outcomes of $\mathcal{S}$ that are not included in A, with $A \cup A^C = \mathcal{S}$

OpenIntro Statistics, 4th Edition