Simple Linear Regression

Grinnell College

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Warm-up

Suppose from a population of male Adelie penguins we take measurements on flipper length and find the following statistics:

$$\overline{x} = 190\text{mm}, \qquad \hat{\sigma} = 6.54\text{mm}$$

If a particular penguin had a standardized flipper length of $z = -0.5$, what was the length of his flipper in millimeters?

Z-scores and Correlation

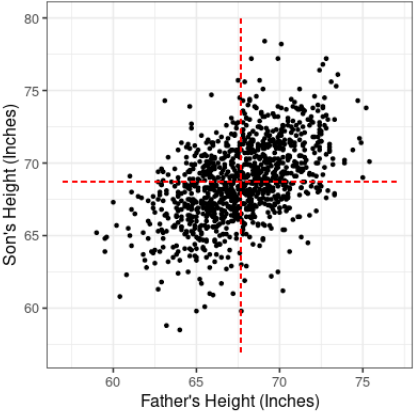
Recall that:

- **Z-scores** or **standardized scores** relate each observation to the mean and standard deviation of the variable
  - $z = 0$ corresponds to the average and $z = 1$ corresponds to one standard deviation
- **Correlation** specifies the *linear* relationship between two quantitative variables

Pearson's Height Data

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

Father	Son
65.0	59.8
63.3	63.2
65.0	63.3
65.8	62.8
61.1	64.3
63.0	64.2
⋮	⋮



Regression towards the mean

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

The correlation coefficient tells us how much "regression" we expect to observe in terms of standardized values:

$$z_S = r \times z_F$$

If the father is one and a half standard deviations above average ($z_F = 1.5$), and the correlation between heights is 0.501, we have:

$$\begin{aligned} z_S &= r \times z_F \\ &= 0.501 \times 1.5 \\ &= 0.752 \end{aligned}$$

Correlation and Prediction

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

From here, we can back substitute the value for z_S to get our unstandardized predictions:

$$\begin{aligned} z_S &= 0.752 \\ \left(\frac{\hat{y} - 68.68}{2.81}\right) &= 0.752 \\ \hat{y} &= 0.752 \times 2.81 + 68.68 \\ \hat{y} &= 70.793 \end{aligned}$$

Where $\hat{y}$ represents our best guess for y, given a value for x

Regression Line

The relationship $z_y = r \times z_x$ can always be manipulated to rewrite the relationship between the variables X and y so they fit the formula

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$

We interpret these as follows:

- $\hat{\beta}_0$ represents the *intercept*, or the estimated value of y when $X = 0$
- $\hat{\beta}_1$ represents the *slope*, indicating the magnitude of change in y given a unit change in X

Regression Line from Z Scores

	Mean (μ)	SD (σ)	Correlation (r_{xy})
Father	67.68	2.74	0.501
Son	68.68	2.81	

Note that $z_F = 1.5$ corresponds to $X = 71.79$

$$z_S = r \times z_F$$
$$\left(\frac{\hat{y} - 68.68}{2.81}\right) = r \times \left(\frac{X - 67.68}{2.74}\right)$$
$$\hat{y} = 33.9 + 0.514X$$

Where $\hat{y}$ represents our best guess for y, given a value for X

Predictions

The formula for the regression line

$$\hat{y} = \beta_0 + X\beta_1$$

can be expressed in terms our our original variables and what we wish to predict

$$\widehat{\text{Son's Height}} = 33.9 + 0.514 \times \text{Father's Height}$$

From this, there are a few things about lines we can observe:

- Using this line, *given* the Father's height, we can predict the son's height using this line by plugging in a value for the father's height
- "For each 1 inch change in Father's height, we expect to see a 0.51 inch change in Son's height"
- Intercept interpretation

Linear Model in R

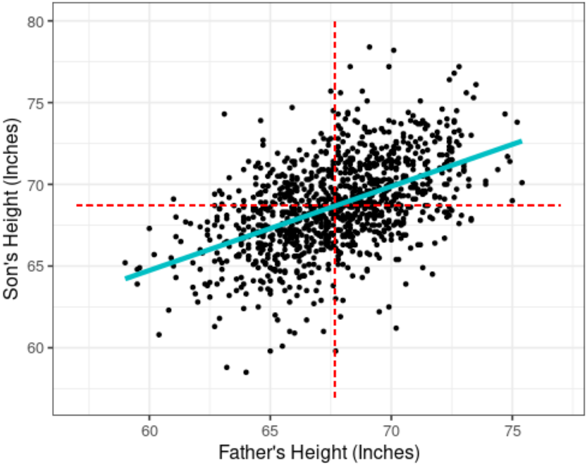
Creating linear models in R is simple; the `lm()` function creates a *linear model* that requires a *formula* component, `Son` $\sim$ `Father` and a data argument, specifying the dataset containing the variables

```
> lm(formula = Son ~ Father, data = dat)

Coefficients:
(Intercept)       Father
     33.893        0.514
```

The output gives us the intercept along with a value for the slope

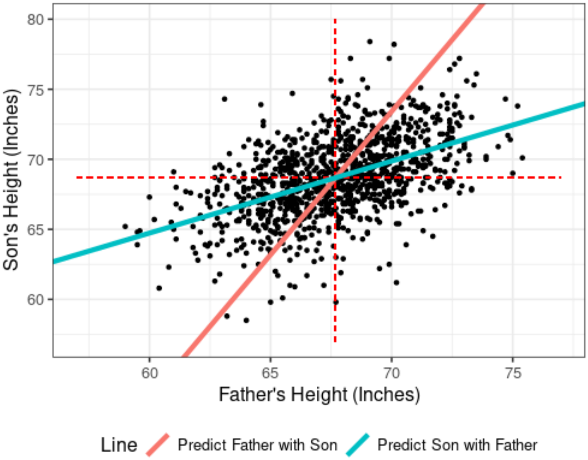
Using Correlation to Make Predictions



"*Given father's height*, the average height of the son is..."

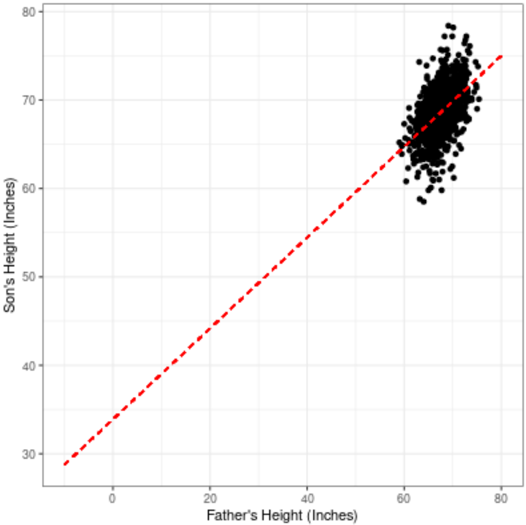
Symmetry

Unlike correlation, where $r_{xy} = r_{yx}$, regression is *asymmetrical*: the choice of explanatory and response variables matter



Intercept Interpretation/Extrapolation

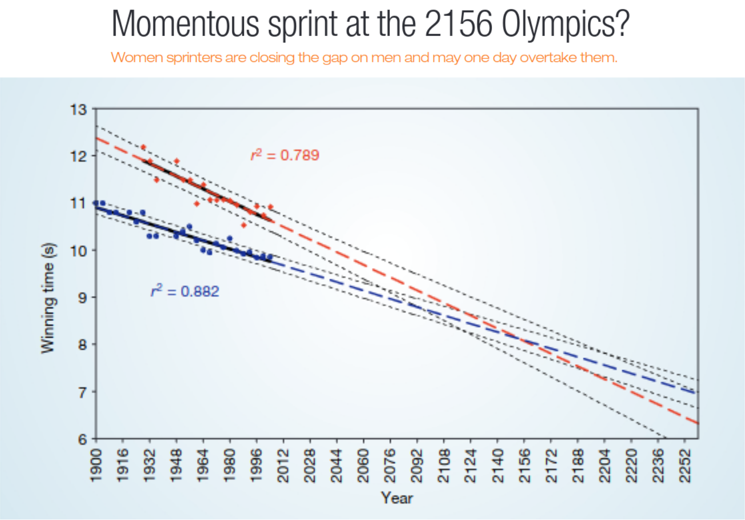
$$\widehat{\text{Son's Height}} = 33.9 + 0.51 \times \text{Father's Height}$$



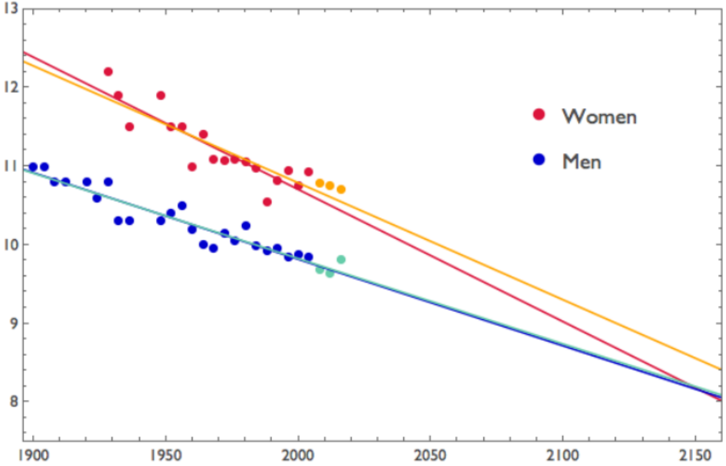
Extrapolation

In 2004, an article was published in *Nature* titled "Momentous sprint at the 2156 Olympics." The authors plotted the winning times of men's and women's 100m dash in every Olympic contest, fitting separate regression lines to each; they found that the two lines will intersect at the 2156 Olympics. Here are a few of the headlines:

- "Women 'may outsprint men by 2156'" – BBC News
- "Data Trends Suggest Women Will Outrun Men in 2156" – Scientific American
- "Women athletes will one day out-sprint men" – The Telegraph
- "Why women could be faster than men within 150 years" – The Guardian

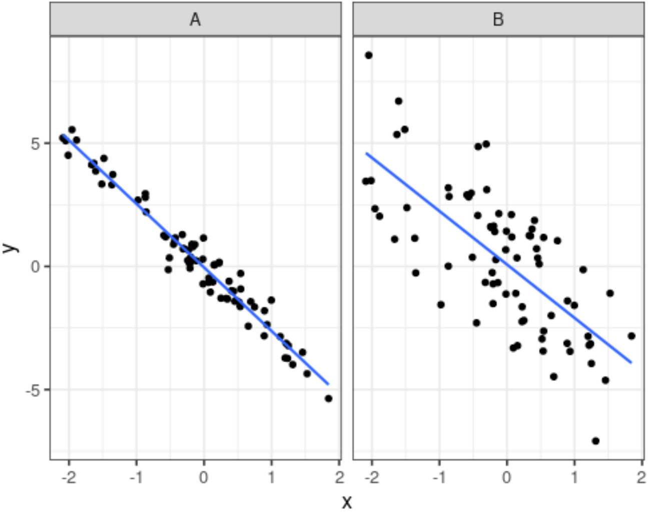

Momentous sprint at the 2156 Olympics?
Women sprinters are closing the gap on men and may one day overtake them.
r^2 = 0.789
r^2 = 0.882
Winning time (s)
Year
13
12
11
10
9
8
7
6
1900 1916 1932 1948 1964 1980 1996 2012 2028 2044 2060 2076 2092 2108 2124 2140 2156 2172 2188 2204 2220 2236 2252

12 years of data later



Assessing Quality of Fit

"How much variability is left once I have selected my prediction on the line?"



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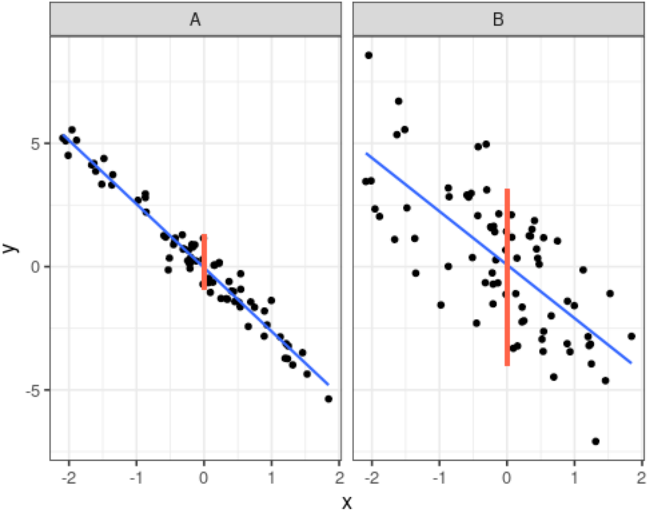


Total Sum of Squares

If we had an outcome y and no predictor variable x, our best guess for an estimate of y would simply by the mean, $\overline{y}$

From this, we get a sense of the *total variance* by taking the *sum of squares*:

$$\text{Total Sum of Squares} = \sum_{i=1}^{n} (y_i - \overline{y})^2$$

We can think of this as our baseline: this is how much variability we see with no other predictors

Regression Sum of Squares

Now assume for each y_i we used a variable x_i, along with their correlation, to create an estimated value $\hat{y}_i$, with

$$\hat{y}_i = \beta_0 + \beta_1 x_i$$

We could then ask ourselves: how much variability is left once I have used my predictor to make $\hat{y}_i$? This gives us the *residual sum of squares*:

$$\text{Residual Sum of Squares} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Coefficient of Determination

Now consider the ratio of variance explained in model against variance without model:

$$\frac{\text{Residual SS (SSR)}}{\text{Total SS (SST)}} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}$$

If our model is no better than guessing the average (i.e., if $\hat{y} = \overline{y}$), this ratio would be 1; if we are able to perfectly predict each value y_i, this ratio would be 0

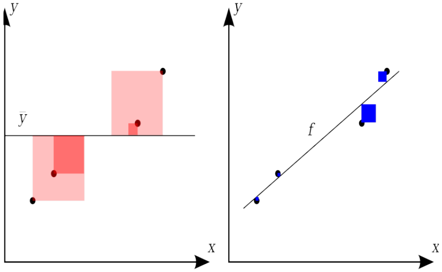
Our **coefficient of determination** or R^2 (R-squared) is defined as

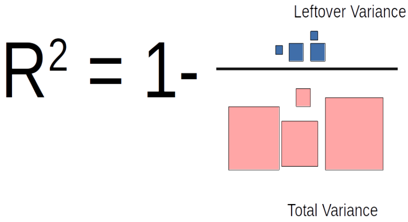
$$R^2 = 1 - \frac{SSR}{SST}$$

Somewhat surprisingly, in the case with a single predictor variable we have that the coefficient of determination is simply the squared correlation

$$R^2 = r^2$$

$$\frac{\text{Residual SS (SSR)}}{\text{Total SS (SST)}} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}$$





Review

We should be able to

- Describe how correlation and regression related
- Be able to predict an outcome, given a predictor
- Interpret the slope and intercept (if applicable)
- Assess the quality of a fitted line