Confidence Intervals II

Grinnell College

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Warm-up

- What is the expression/formula for a confidence interval?
- What factors can change the length of my interval? How do they change it?
- What are some things I should consider when deciding how big to make my interval?

Review

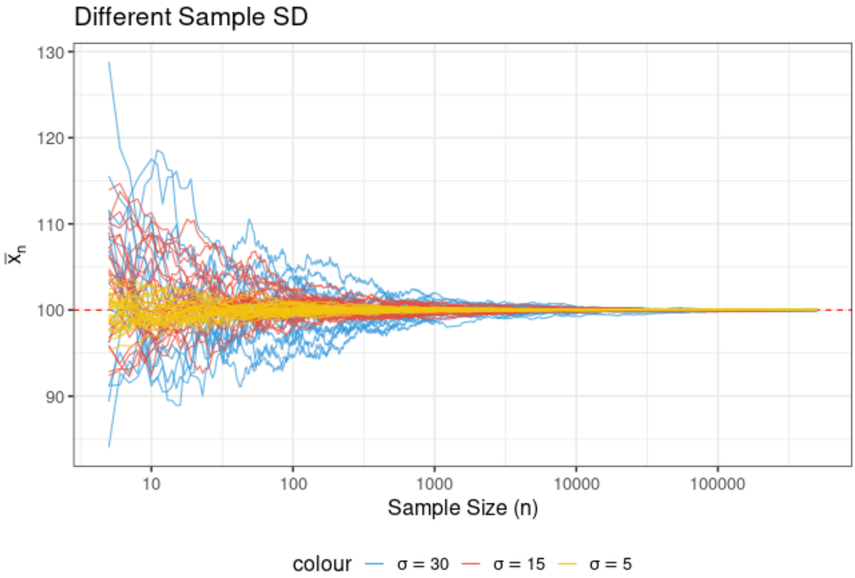
The **Law of Large Numbers** guarantees that, as the number of observations n in my sample increases, my estimate of the parameter will converge to the true value

A **sampling distribution** refers to the distribution of a sample statistic (i.e., $\overline{x}$) if we were to repeatedly sample from a population and recompute the statistic

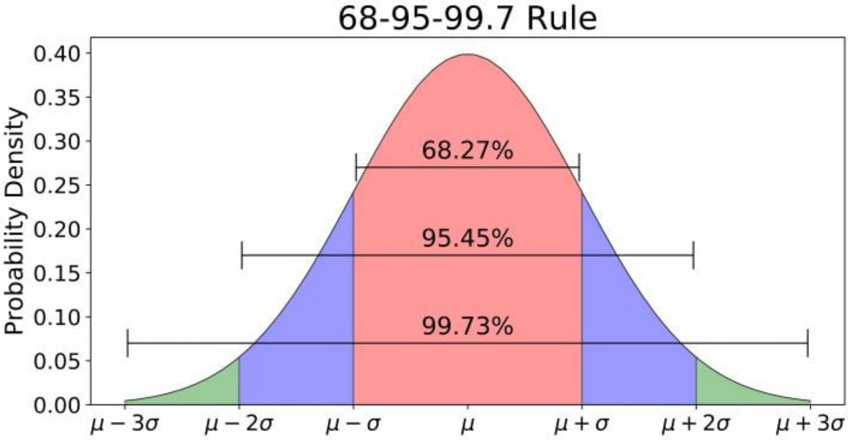
- What values would they take?
- How frequently would they appear?

The **Central Limit Theorem** states that is my statistic is an average or a proportion, then the sampling distribution of my statistic will be approximately normal, with

$$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$


Different Sample SD
130
120
110
100
90
$\bar{x}_n$
10
100
1000
10000
100000
Sample Size (n)
colour
σ = 30
σ = 15
σ = 5

Empirical Rule



Point $\pm$ Margin of Error

The key idea is this:

In finding a plausible range for our parameter μ, we want to create an interval, centered at our observed statistic, that takes the form

$$\overline{X} \pm C \times \left(\frac{\hat{\sigma}}{\sqrt{n}} \right)$$

Our **confidence**, mediated by C, is the frequency with which an interval constructed in this way will contain the true value of the mean.

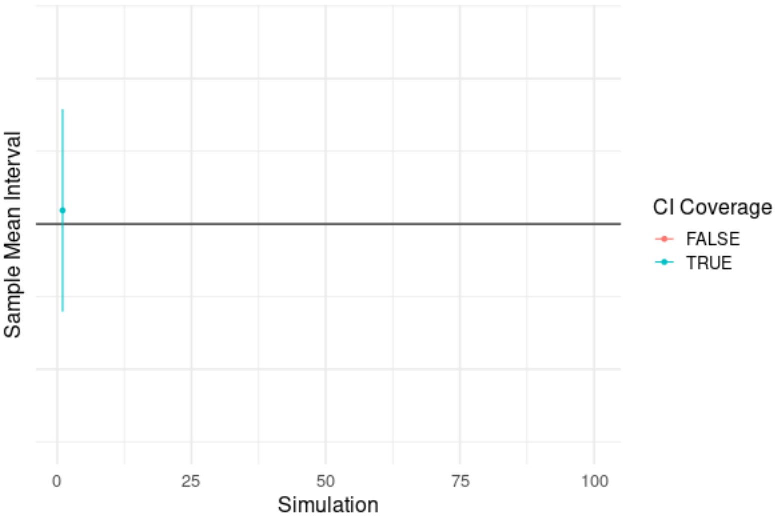
Critical Values and Percentiles

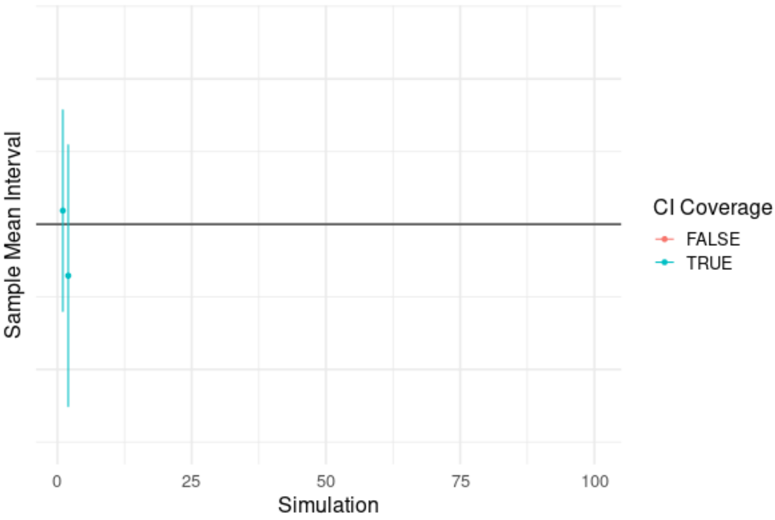
The C term in the expression

$$\overline{X} \pm C \times \left(\frac{\hat{\sigma}}{\sqrt{n}} \right)$$

is called a **critical value**. The choice of C is *the only thing we choose that determines the confidence level of our interval*

Put another way: there are several things that determine the *length* of an interval, but only one that determines its confidence


Sample Mean Interval
0
25
50
75
100
Simulation
CI Coverage
FALSE
TRUE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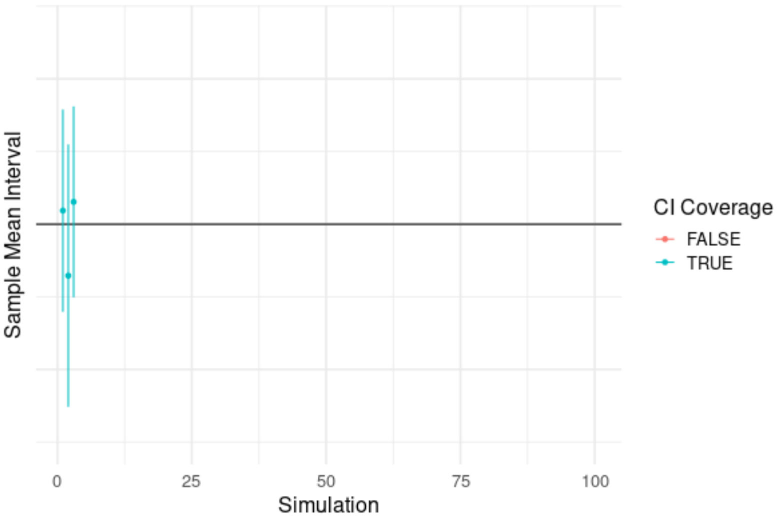
75
100
Simulation
CI Coverage
FALSE
TRUE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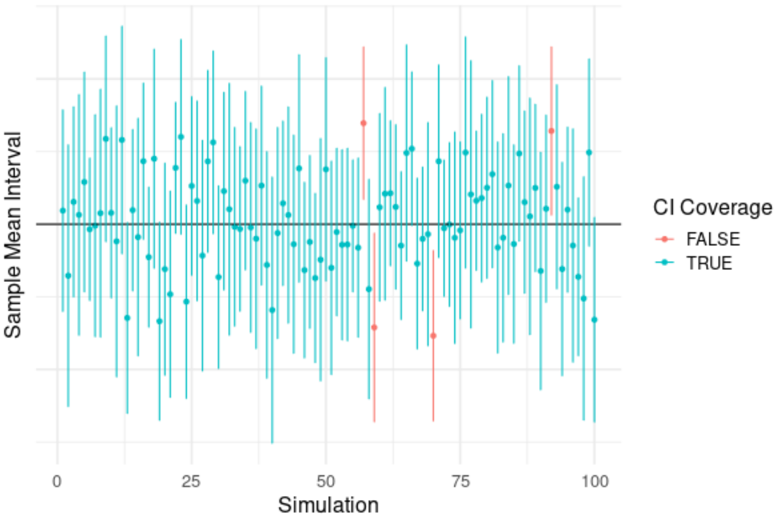
TRUE



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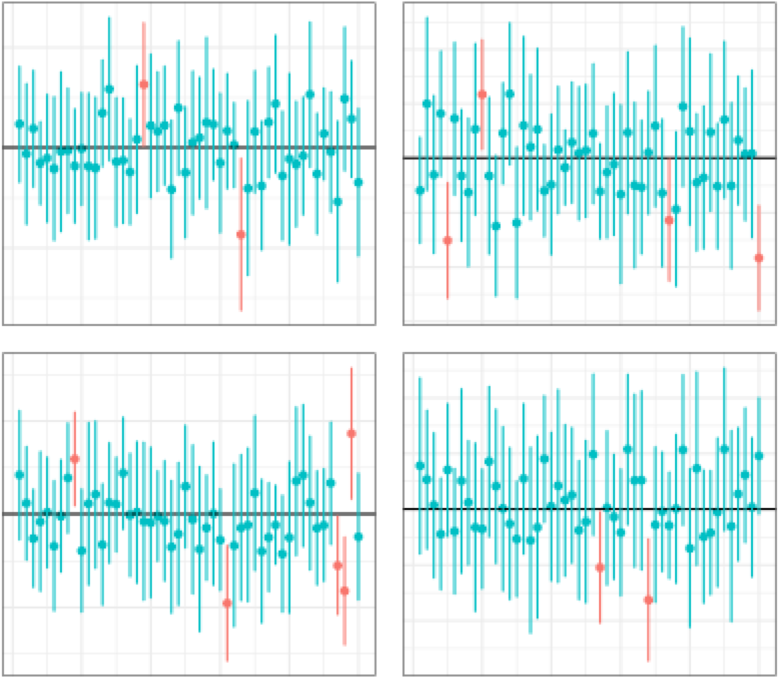
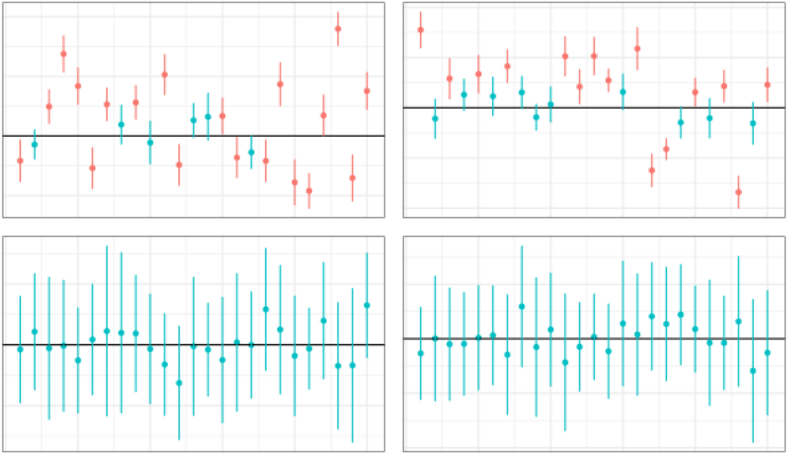
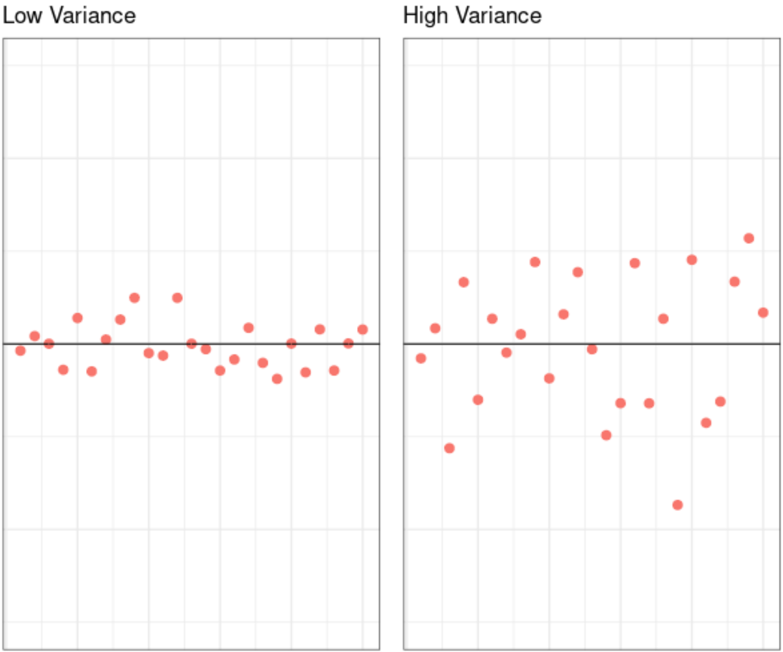
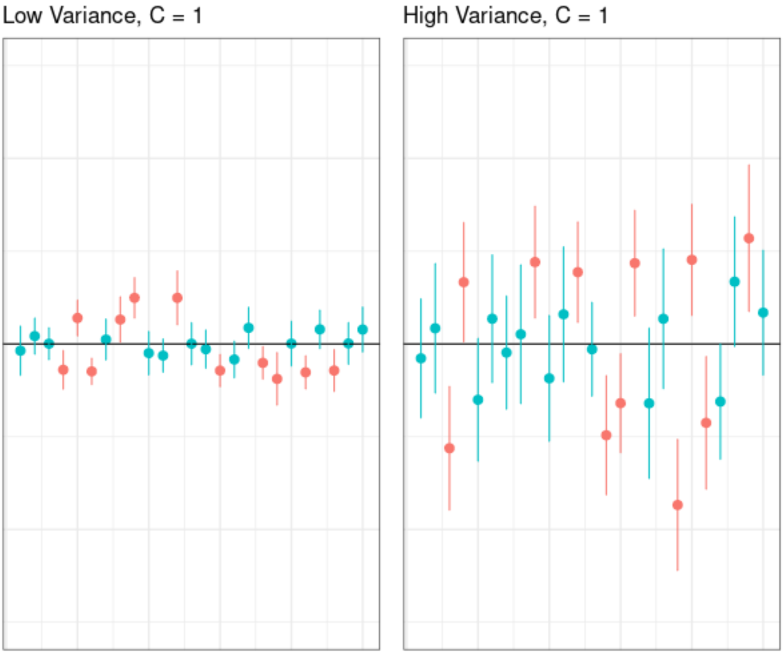


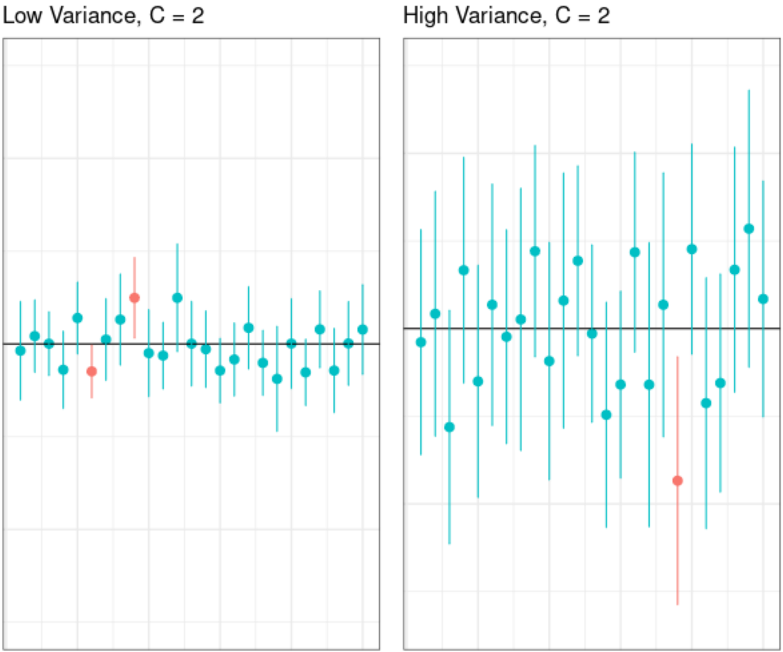
Confidence Intervals

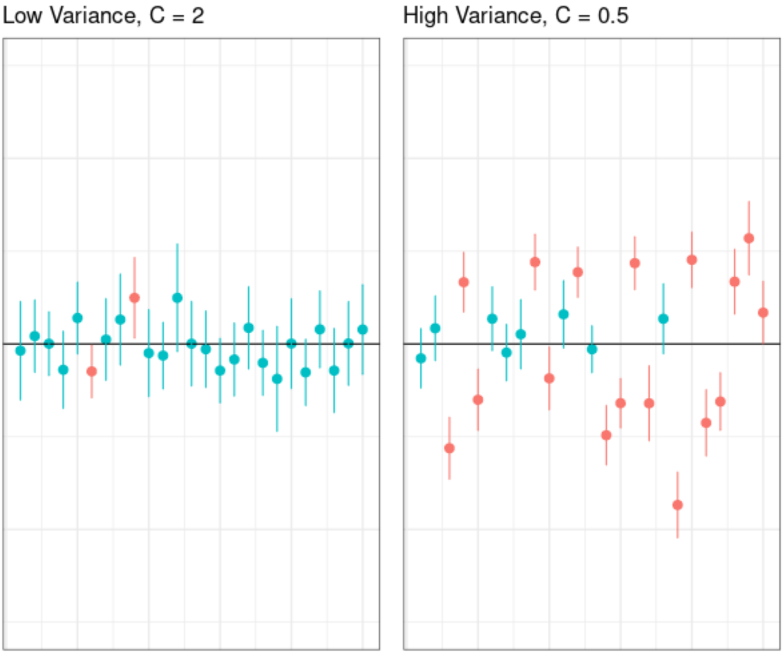
It is also worth observing that we can *alter* our process to achieve different results. There is a trade-off between how frequently we are correct and how much uncertainty we allow in our prediction

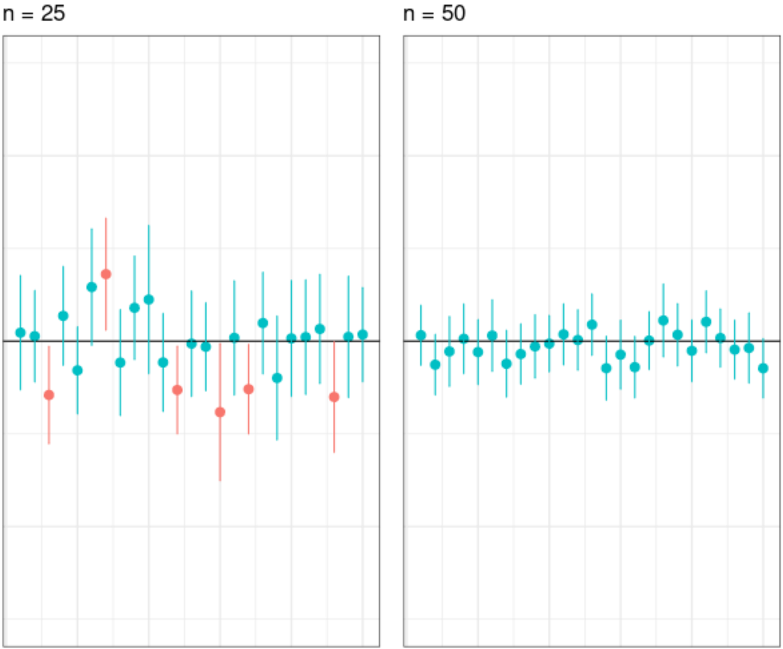



Low Variance
High Variance


Low Variance, C = 1
High Variance, C = 1


Low Variance, C = 2
High Variance, C = 2


Low Variance, C = 2
High Variance, C = 0.5


n = 25
n = 50

Examples

Suppose I sample $n = 20$ male Adelie penguins from a population where $\mu = 40.39$ and $\sigma = 2.2771$. From my sample, I find the statistics,

$$\overline{x} = 40.63, \quad \hat{\sigma} = 1.97, \quad \frac{\hat{\sigma}}{\sqrt{n}} = 0.44$$

Find confidence intervals associated with the critical values for:

- $C = 0.5$
- $C = 1$
- $C = 3$

Review

- **Standard deviation** (σ) is an estimate of the amount of variability in our sample, while **standard error** ($\sigma/\sqrt{n}$) is an estimate of the variability in estimating a parameter
- A **sampling distribution** describes the distribution of a statistic or parameter estimate if we could repeat the sampling process as many times as we wish
- Approximations to the normal distribution generally follow the **66-95-99 rule** with $1/2/3$ standard deviations of the mean
- If these properties hold, we can create a reasonable interval of possible parameter values of the form Point Estimate $\pm$ Margin of Error
- A **confidence interval** is an interval with the properties that:
  - It is constructed according to a procedure or set of rules
  - It is intended to give plausible range of values for a *parameter* based on a *statistic*
  - It has no probability; the interval either contains the true value or it does not