Residuals

Grinnell College

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Review

Below is a model of chicken weight in days since birth according to one of four separate diets

```
> lm(weight ~ Time + Diet, ChickWeight) %>% summary()

            Estimate Std. Error t value             Pr(>|t|)
(Intercept)   10.924      3.361    3.25               0.0012 **
Time           8.750      0.222   39.45 < 0.0000000000000002 ***
Diet2         16.166      4.086    3.96     0.00008556049098 ***
Diet3         36.499      4.086    8.93 < 0.0000000000000002 ***
Diet4         30.233      4.107    7.36     0.00000000000064 ***

Multiple R-squared:  0.745, Adjusted R-squared:  0.744
F-statistic:  419 on 4 and 573 DF,  p-value: <0.0000000000000002
```

1. Write out the equation for this linear model
2. What proportion of the total variability in chick weight is described by this model?
3. Which diet resulted in the heaviest weight? The lightest?
4. How much would you predict a chicken on Diet 1 weighs after 15 days?
5. After 10 days, what is the predicted difference in weight between a chicken on Diet 2 and Diet 3?

Today

- Regression posits linear relationship between dependent variable y and independent variable X of the form

$$y = \beta_0 + \beta_1 X + \epsilon$$

- Expand this to include combinations of independent variables, both qualitiative and quantitative
- Today our focus is on the error term ϵ (epsilon)

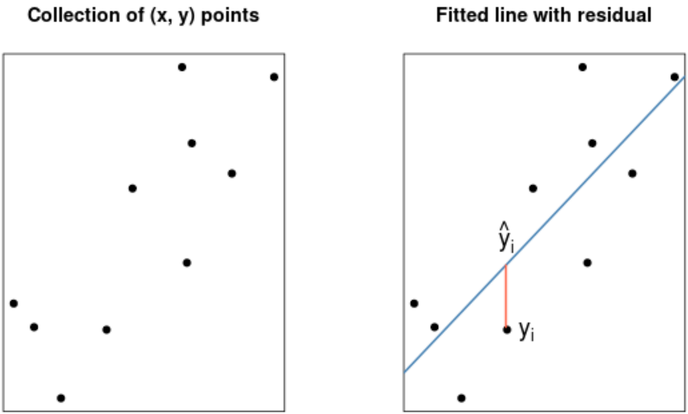
Error Terms

$$y = \beta_0 + X\beta_1 + \epsilon$$

Assumptions:

- Linear relationship between X and y
- Error term is normally distributed, $\epsilon \sim N(0, \sigma)$
- Variability in error should be the same for all values of X, i.e., error same for all observations

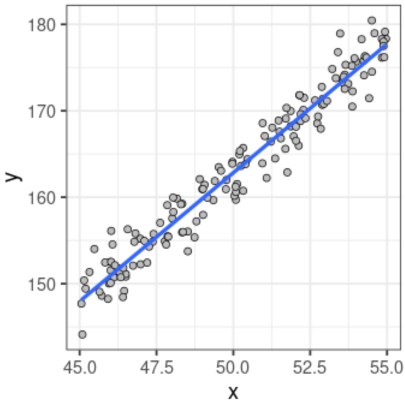
Analyzing the error terms gives us a way to test the assumptions of our model

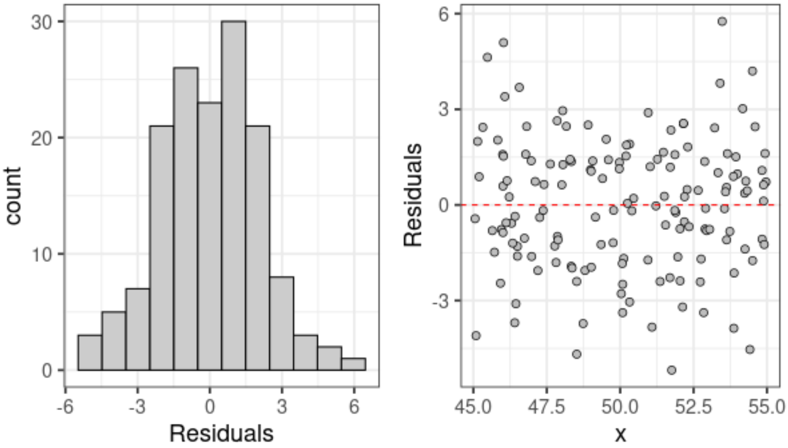

Collection of (x, y) points
Fitted line with residual
ŷi
yi

Part 1: Checking Assumptions

Three common ways to investigate residuals visually:

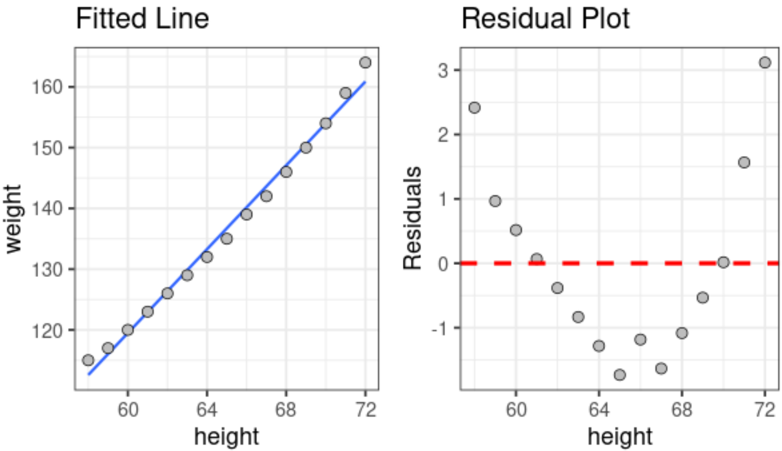
1. Plot histogram of residuals (normality)
2. Plot residuals against covariate (linear trend, homoscedasticity)
3. Plot residuals against new covariates (pattern identification)





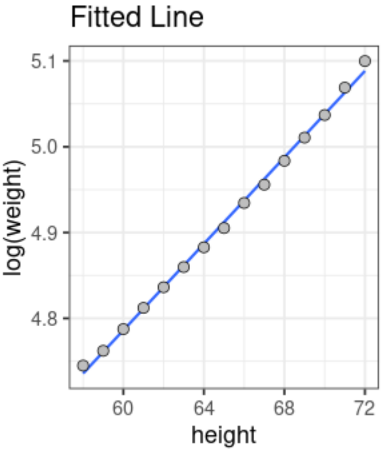
30
20
10
0
count
-6
-3
0
3
6
Residuals
6
3
0
-3
Residuals
45.0
47.5
50.0
52.5
55.0
x

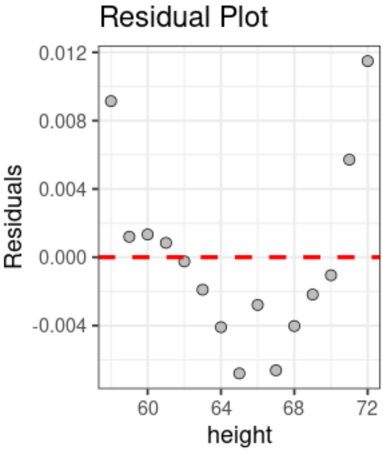
Tests of linearity



Tests of linearity

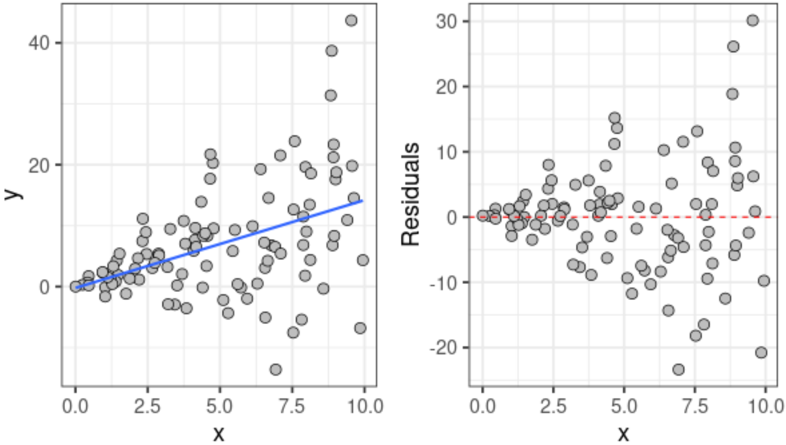
Sometimes a transformation of a variable (in this case, log(weight)) can help correct trends





Heteroscedasticity

Hetero = different, scedastic = random



Part 2: Investigating Patterns

Correlated Covariates

Consider a simple linear model in which a covariate X is used to predict some value y

$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$

The residuals associated with this describe the amount of variability that *is yet to be explained*

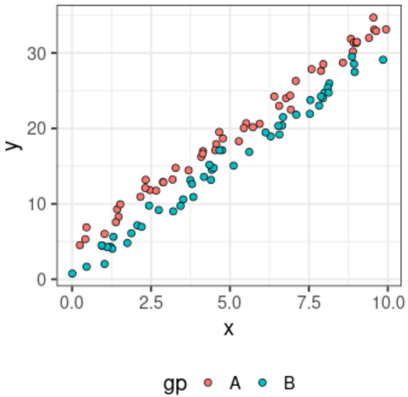
$$r = \hat{y} - y$$

The idea is to find new covariates *associated* with this residual, in effect "mopping up" the remaining uncertainty

Considering new covariates (categorical)

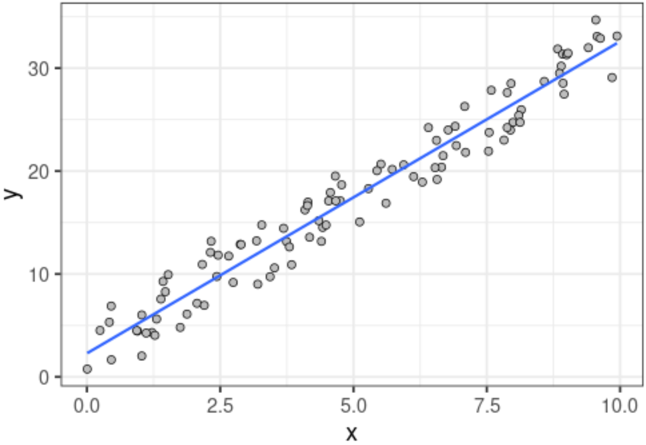
Suppose I have:

- Quantitative outcome *y*
- Quantitative predictor *X*
- Categorical predictor *gp*

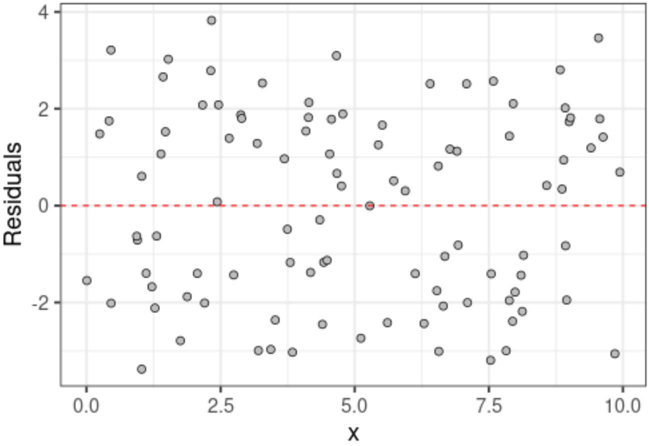


Considering new covariates (categorical)

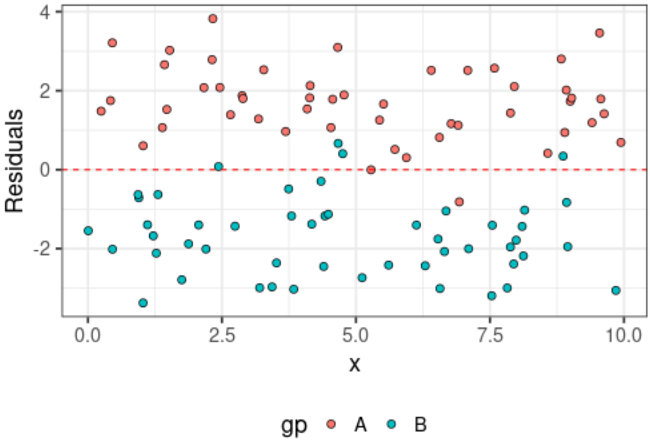
$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1$$



Considering new covariates (categorical)

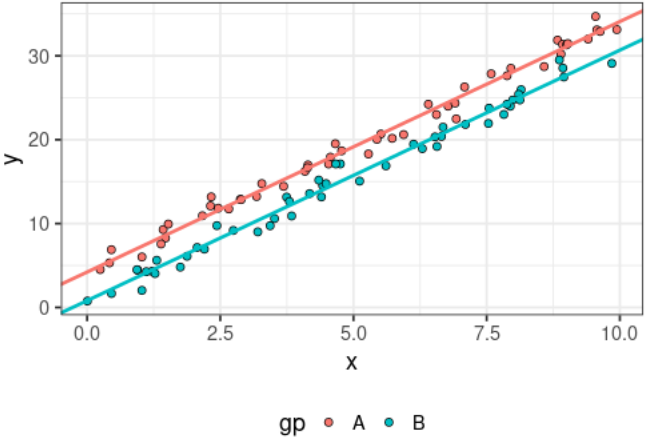


Considering new covariates (categorical)

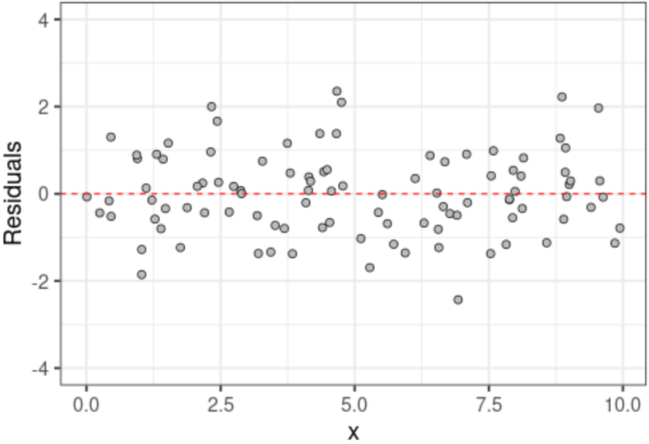


Considering new covariates (categorical)

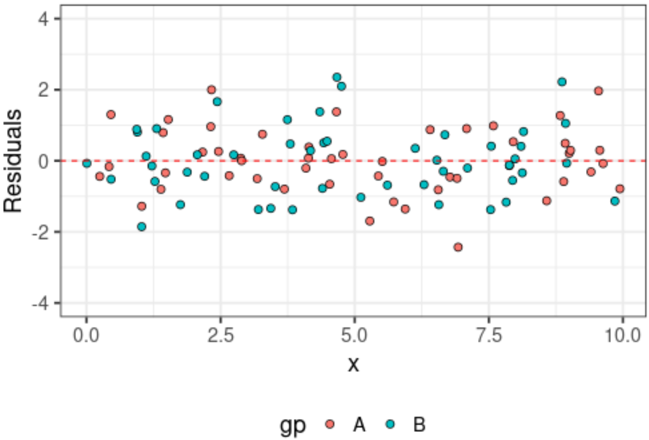
$$\hat{y} = \hat{\beta}_0 + X\hat{\beta}_1 + \mathbb{1}_{gp}\hat{\beta}_2$$



Considering new covariates (categorical)



Considering new covariates



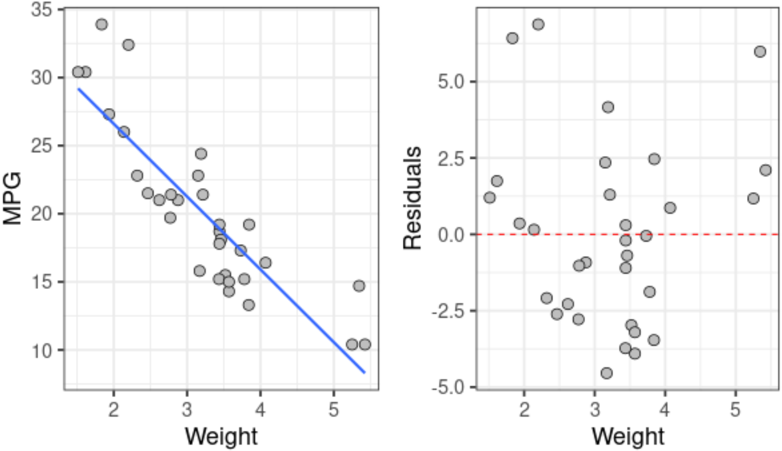
Considering new covariates (quantiative)

Now consider a situation in which we wish to predict fuel economy with three separate models:

1. Using weight
2. Using weight and engine displacement
3. Using weight and quarter mile time

Starting with the first model, we can consider the relationship of the residuals with both engine displacement and quarter mile time

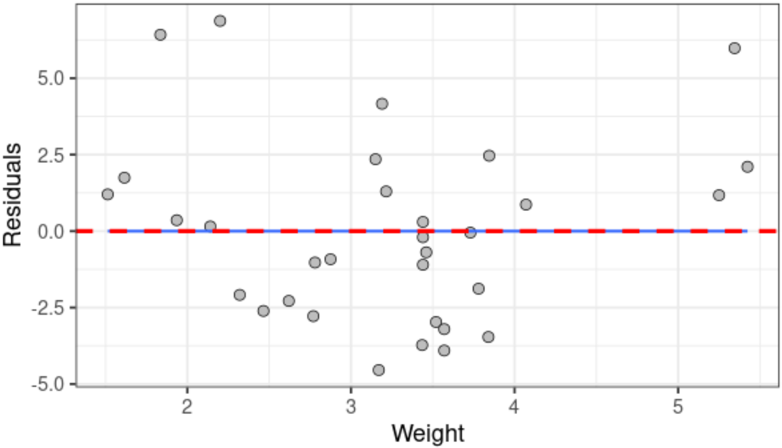
Considering new covariates (quantiative)



Considering new covariates (quantiative)

An interesting thing occurs when we try to create a regression model of the residuals with the original variable:

$$\text{residuals} = \hat{\beta}_0 + \hat{\beta}_1 \text{Weight} = 0$$



Considering new covariates (quantiative)

When considering adding new variables to our regression model, we want to add those that will "mop up" the residuals that are left after considering weight

This brings us to the idea of **correlated variables**, or variables that have evidence of a *linear relationship* with one another

Models that have a large number of correlated variables suffer from issues of **multicollinearity**, a phenomenon we will briefly investigate

Correlated Covariates

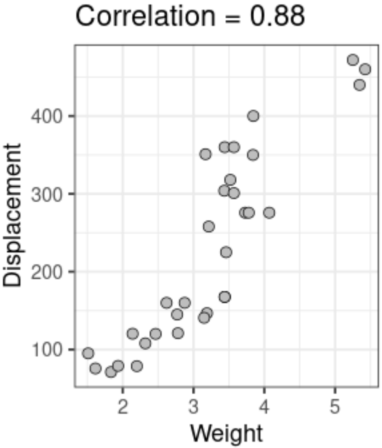
We can consider two extremes: if two quantitative variables are perfectly correlated, knowing the value of one variable means we also know the value of the other

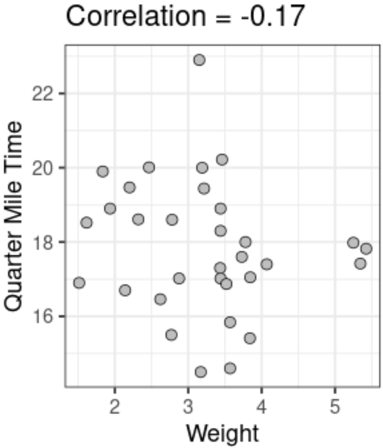
This means that, in terms of predicting an outcome, adding a highly correlated variable to our model will contribute little new information and will not be very useful

By constrast, if two variables have perfectly uncorrelated (i.e., $\rho = 0$), then knowing the value of one tells us nothing about the value of another

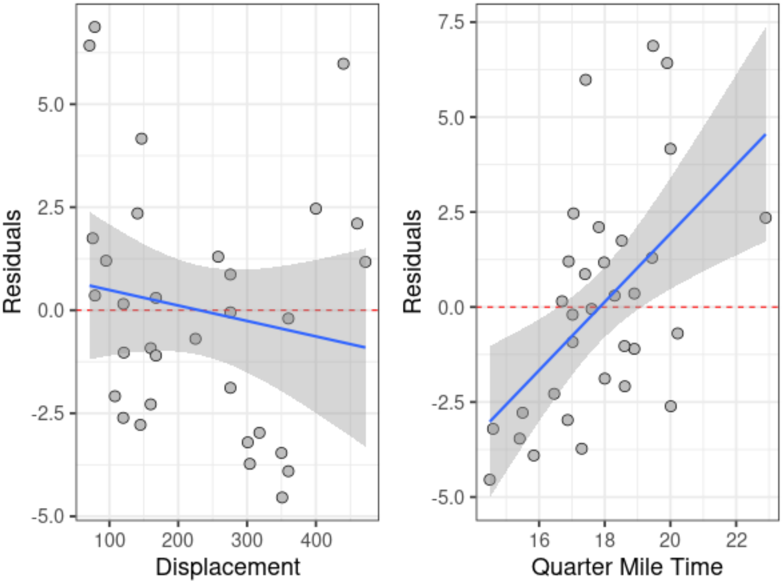
Adding an uncorrelated variable to our model thus offers more potential to "mop up" the variability that was not explained by the first variable

Correlated Covariates





Residual Plots



Correlated Covariates

```
> lm(mpg ~ wt, mtcars) %>% summary()

             Estimate Std. Error t value     Pr(>|t|)
(Intercept)    37.285      1.878   19.86   < 0.000002 ***
wt             -5.344      0.559   -9.56     0.000013 ***
R-squared = 0.75

> lm(mpg ~ wt + disp, mtcars) %>% summary()

             Estimate Std. Error t value      Pr(>|t|)
(Intercept) 34.96055    2.16454   16.15   0.000000049 ***
wt          -3.35083    1.16413   -2.8         0.0074 **
disp        -0.01772    0.00919   -1.93        0.0636 .
R-squared = 0.78

> lm(mpg ~ wt + qsec, mtcars) %>% summary()

             Estimate Std. Error t value        Pr(>|t|)
(Intercept)    19.746      5.252    3.76         0.00077 ***
wt             -5.048      0.484  -10.43 0.000000000025 ***
qsec            0.929      0.265    3.51         0.00150 **
R-squared = 0.82
```

Key Takeaways

1. Number of assumptions for linear model
   - Linearity
   - Normal errors
   - Homoscedasticity
2. Need way to determine which new variables to add to model
3. Examining errors effective way to test assumptions and investigate new covariates
4. Relationship between correlation of predictors and residual analysis