Probability (Part 2)

Grinnell College

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Review, what did we do last time

$$P(A \text{ or } B) = P(A) + P(B) + P(A \text{ and } B)$$

We didn't really discuss how to find $P(A \text{ and } B)$

Independence

We say that two *random processes* are **independent** if the outcome of one process provides no information about the outcome of another

Examples include:

- Flipping a coin multiple times
- Rolling a red and white dice together
- Sampling different colored marbles from a jar *with* replacement

Independence

If two random processes A and B two different and *independent* processes, then the probability that both A and B occur is

$$P(A \text{ and } B) = P(A) \times P(B).$$

This is known as the **Multiplication Rule**

Board work:

- Coins
- Marbles
- Trees

Consider the results of a machine learning algorithm that was tasked with classifying 1600 photos as either being a puppy or not being a puppy. A tabulation of the actual photos, along with their predictions, is included in the table below

	Actual Photo		
	Puppy	Not Puppy	Total
Classified Puppy	500	100	600
Classified Not Puppy	200	800	1000
Total	700	900	1600

	Actual Photo		
	Puppy	Not Puppy	Total
Classified Puppy	500	100	600
Classified Not Puppy	200	800	1000
Total	700	900	1600

From this, we may ask ourselves: what is the probability that a randomly selected photo is of a puppy?

$$P(\text{puppy}) = \frac{\text{\# of puppy photos}}{\text{Total \# of photos}} = \frac{700}{1600} = 0.43$$

Or we may ask: what is the probability that a randomly selected photo is of a puppy and has been classified as a puppy?

$$\begin{aligned} P(\text{Is puppy and classified puppy}) &= \frac{\text{\# of puppy photos classified as puppy}}{\text{Total \# of photos}} \\ &= \frac{500}{1600} = 0.31 \end{aligned}$$

This brings us to a trio of definitions:

The **marginal probability** of a sample describes the probability of a *single* variable without regard to others, i.e., the probability of event A is $P(A)$

The **joint probability** of a sample describes the probabilities for two or more outcomes together, i.e., the probability of events A and B both is $P(A \text{ and } B)$

Conditional probability describes the probability of one event based on the assumed outcome of another, i.e., the conditional probability of event A *given* B is denoted $P(A|B)$

	Actual Photo		
	Puppy	Not Puppy	Total
Classified Puppy	500	100	600
Classified Not Puppy	200	800	1000
Total	700	900	1600

Does knowing how a photo was classified change our estimate of the probability that a given photo was of a puppy?

$$\begin{aligned} P(\text{Is puppy } \textit{given} \text{ classified puppy}) &= \frac{\#\text{ of puppy photos classified as puppy}}{\text{Total }\#\text{ classified as puppy}} \\ &= \frac{500}{600} \\ &= 0.83 \end{aligned}$$

However, we can also compute this same thing from *marginal* and *joint* probabilities

	Actual Photo		
	Puppy	Not Puppy	Marginal Prob
Predict Puppy	0.312	0.062	0.375
Predict Not Puppy	0.125	0.500	0.625
Marginal Prob	0.438	0.562	1

$$\begin{aligned} P(\text{Is puppy } \textit{given} \text{ classified puppy}) &= \frac{P(\text{Is puppy and classified puppy})}{P(\text{Classified puppy})} \\ &= \frac{0.312}{0.375} \\ &= 0.83 \end{aligned}$$

Conditional Probability

From this, we have the following relation:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Rewriting this gives us the **General Multipliction Rule**:

$$P(A \text{ and } B) = P(A|B) \times P(B)$$

Note that when A and B are independent, $P(A|B) = P(A)$, giving us our original **Multipliction Rule**:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Clear up

We can take this machinery and return to some of our examples from Monday

$$S = \{1, 2, 3, 4, 5, 6\}$$

Consider:

- $A = \{1, 2, 3, 5\}$
- $B =$ roll a number greater than four

Can we find:

- $P(A)$
- $P(B)$
- $P(A|B)$
- $P(B|A)$
- $P(A \text{ and } B)$

OpenIntro Statistics, 4th Edition